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Matrix trace Wielandt inequalities with statistical applications

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ABSTRACT

The vector correlation coefficient and other measures of association play a very important role in statistics and especially in multivariate analysis. In this paper a new measure of association is proposed and its upper bound is presented by using a matrix trace Wielandt inequality. Also given are relevant results involving Wishart matrices widely used in multivariate analysis, and especially a new alternative for the relative gain of the covariance adjusted estimator of a vector of parameters.

1. Introduction

It seems that the Wielandt inequality (WI) in the vector case was introduced by Bauer and Householder (1960) due to a private communication from Wielandt; see Drury et al. (2002, Section 2). Let $\mathbf{A}$ be a positive definite symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$, and let $\mathbf{x}$ and $\mathbf{y}$ be two nonnull real vectors satisfying $\mathbf{x}'\mathbf{y} = 0$. Then

$$\frac{(\mathbf{x}'\mathbf{A}\mathbf{y})^2}{\mathbf{x}'\mathbf{A}\mathbf{x} \cdot \mathbf{y}'\mathbf{A}\mathbf{y}} \leqslant \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2. \tag{1.1}$$

We will refer to (1.1) as the "WI". The first appearance of (1.1) in a statistical context seems to be by Eaton (1976). Let the random vector $\mathbf{h}$ have the covariance matrix $\mathbf{A}$; then the maximum of the squared correlation

$$\max_{\mathbf{x},\mathbf{y}:\mathbf{x}'\mathbf{y}=0} \operatorname{corr}^2(\mathbf{x}'\mathbf{h}, \mathbf{y}'\mathbf{h}) = \max_{\mathbf{x},\mathbf{y}:\mathbf{x}'\mathbf{y}=0} \frac{(\mathbf{x}'\mathbf{A}\mathbf{y})^2}{\mathbf{x}'\mathbf{A}\mathbf{x} \cdot \mathbf{y}'\mathbf{A}\mathbf{y}} = \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2.$$

It is known that the WI can be viewed as a constrained version of the Cauchy–Schwarz inequality (CSI), which links with the Frucht–Kantorovich inequality (FKI) in a nice way. We remind the reader about the FKI which can be expressed as follows:

$$\frac{\mathbf{x}'\mathbf{A}\mathbf{x} \cdot \mathbf{x}'\mathbf{A}^{-1}\mathbf{x}}{(\mathbf{x}'\mathbf{x})^2} \leqslant \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}.$$

For matrix, determinant and trace versions of the CSI and the FKI, see, e.g., Liu (1995, 1999, 2000), Rao and Rao (1998), Magnus and Neudecker (1999), Zhang (1999) and Liu and Heyde (2003). For a survey of matrix CSIs and FKIs, see Liu and Neudecker

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