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DECOMPOSING MATRICES WITH JERZY K. BAKSALARY

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Abstract

In this paper we comment on some papers written by Jerzy K. Baksalary. In particular, we draw attention to the development process of some specific research ideas and papers now that some time, more than 15 years, has gone after their publication.

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1. INTRODUCTION

In the Acknowledgements of Simo Puntanen's Ph.D. Thesis (1987), there was one sentence concerning Jerzy K. Baksalary: "The kind comments of Dr. Jerzy K. Baksalary helped to improve Section 3.6 of the first paper [of the thesis]." It is interesting to go back in time 20 years and refresh the memory of what was going on then.

The Section 3.6 of the thesis begins as follows:

> "In this section we consider various representations for the BLUE's covariance matrix; some of these representations are well known. In particular, we will study the effect of the condition $u = 0$, i.e., there are no unit canonical correlations between $\mathbf{Hy}$ and $\mathbf{My}$. We believe that our results on the effects of this condition on the BLUE's covariance matrix are new. Some of the following formulae were introduced by Puntanen (1986)."

The header of Section 3.6 is "BLUE's covariance matrix". This was a very important section of the thesis and interestingly, it seems to have served as a kind of seed for several papers dealing with related problems.

In Section 2 of this paper we go through some developments originated from Section 3.6 of Puntanen's dissertation. In Section 3 we shall take a quick look at some results concerning the matrix $\dot{\mathbf{M}}$, which is strongly related to a special case of a decomposition introduced by Baksalary, Puntanen and Styan (1990). And finally, in Section 4, we consider the concept of linear sufficiency, which appeared to be a crucial concept for Jarkko Isotalo (2007) in his recent dissertation.

Before jumping into Section 3.6 of Puntanen (1987), a couple of clarifying remarks about the notation may take a place. Throughout the paper we consider the general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1.1}$$

briefly denoted as $\mathcal{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \sigma^2\mathbf{V}\}$, where

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}, \quad \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \mathrm{cov}(\mathbf{y}) = \mathrm{cov}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{V}. \tag{1.2}$$

By $\mathrm{E}(\cdot)$ and $\mathrm{cov}(\cdot)$ we denote expectation vector and covariance matrix of a random vector argument. The vector $\mathbf{y}$ is an $n \times 1$ observable random vector, $\boldsymbol{\varepsilon}$ is an $n \times 1$ random error vector, $\mathbf{X}$ is a known $n \times p$ model matrix, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, $\mathbf{V}$ is a known $n \times n$ nonnegative definite matrix, and σ^2 is an unknown nonzero constant. If the scalar σ^2 plays no role we assume that $\sigma^2 = 1$.

The symbols $\mathbf{A}'$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})^\perp$, $\mathscr{N}(\mathbf{A})$, and $\mathrm{r}(\mathbf{A})$ will denote, respectively, the transpose, a generalized inverse, the Moore–Penrose inverse, the column space, the orthogonal complement of the column space, the null space, and the rank of the matrix $\mathbf{A}$. Furthermore, we will write $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector (with respect to the standard inner product) onto $\mathscr{C}(\mathbf{A})$. In particular, we will denote

$$\mathbf{H} = \mathbf{P}_\mathbf{X}, \quad \mathbf{M} = \mathbf{I} - \mathbf{H}, \tag{1.3}$$

and so the ordinary least squares (OLS) estimator of $\mathbf{X}\boldsymbol{\beta}$ is

$$\mathrm{OLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{H}\mathbf{y} = \mathbf{P}_\mathbf{X}\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} = \widehat{\mathbf{X}\boldsymbol{\beta}}, \tag{1.4}$$

and the corresponding vector of residuals is $\mathbf{e} = \mathbf{y} - \mathbf{H}\mathbf{y} = \mathbf{M}\mathbf{y}$.

An unbiased estimator $\mathbf{G}\mathbf{y}$ is the best linear unbiased estimator (BLUE) of $\mathbf{X}\boldsymbol{\beta}$ if

$$\mathbf{G}\mathbf{V}\mathbf{G}' \leq_\mathrm{L} \mathbf{B}\mathbf{V}\mathbf{B}' \quad \forall\, \mathbf{B} : \mathbf{B}\mathbf{X} = \mathbf{X}, \tag{1.5}$$

where "$\leq_\mathrm{L}$" refers to the Löwner ordering. We will use the notation

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}\tilde{\boldsymbol{\beta}} = \widetilde{\mathbf{X}\boldsymbol{\beta}}. \tag{1.6}$$

When $\mathbf{V}$ is nonsingular then the BLUE of $\mathbf{X}\boldsymbol{\beta}$ is

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}. \tag{1.7}$$

We may recall here three general representations for the BLUE($\mathbf{X}\boldsymbol{\beta}$):

$$\text{(1.8a)} \qquad \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{Hy} - \mathbf{HVM}(\mathbf{MVM})^{-}\mathbf{My} := \mathbf{G}_1\mathbf{y},$$

$$\text{(1.8b)} \qquad \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{y} - \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{My} := \mathbf{G}_2\mathbf{y},$$

$$\text{(1.8c)} \qquad \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}^{-}\mathbf{y} := \mathbf{G}_3\mathbf{y},$$

where $\mathbf{W}$ (and the related $\mathbf{U}$) are any matrices such that

$$\text{(1.9)} \qquad \mathbf{W} = \mathbf{V} + \mathbf{XUX}', \quad \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}).$$

2. BLUE's covariance matrix

2.1. Developments by 1987

In this section we go through (not in details) some results of Section 3.6 of Puntanen's dissertation (1987) and then describe how the research thereby initiated developed further. The main articles of interest in this context are Baksalary, Puntanen and Styan (1990) and Isotalo, Puntanen and Styan (2007).

Let us now return back to Section 3.6 of Puntanen (1987) where the following representations for the BLUE's covariance matrix are given:

- General case:

$$\text{(2.1a)} \qquad \mathrm{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{HVH} - \mathbf{HVM}(\mathbf{MVM})^{-}\mathbf{MVH}$$

$$\text{(2.1b)} \qquad = \mathbf{V} - \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{MV}$$

$$\text{(2.1c)} \qquad = \mathbf{V} - \mathbf{V}\dot{\mathbf{M}}\mathbf{V}$$

$$\text{(2.1d)} \qquad = \mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}\mathbf{X}' - \mathbf{XUX}',$$

where

$$\dot{\mathbf{M}} = \mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}, \tag{2.2}$$

and $\mathbf{U}$ and $\mathbf{W}$ are defined as in (1.9).

- $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{V})$ (i.e., the model is weakly singular):

$$\operatorname{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-}\mathbf{X})^{-}\mathbf{X}' \tag{2.3a}$$

$$= \mathbf{X}_b(\mathbf{X}_b'\mathbf{V}^{-}\mathbf{X}_b)^{-}\mathbf{X}_b' \tag{2.3b}$$

$$= \mathbf{H}(\mathbf{H}\mathbf{V}^{-}\mathbf{H})^{-}\mathbf{H} \tag{2.3c}$$

$$= (\mathbf{H}\mathbf{V}^{-}\mathbf{H})^{+}. \tag{2.3d}$$

- $\mathbf{V}$ is positive definite:

$$\operatorname{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}' \tag{2.4a}$$

$$= \mathbf{X}_b(\mathbf{X}_b'\mathbf{V}^{-1}\mathbf{X}_b)^{-1}\mathbf{X}_b' \tag{2.4b}$$

$$= \mathbf{H}(\mathbf{H}\mathbf{V}^{-1}\mathbf{H})^{-}\mathbf{H} \tag{2.4c}$$

$$= (\mathbf{H}\mathbf{V}^{-1}\mathbf{H})^{+}, \tag{2.4d}$$

where $\mathbf{X}_b$ is a matrix whose columns form a basis for $\mathscr{C}(\mathbf{X})$.

We may note that the matrix $\mathbf{G}$ having the property $\mathbf{Gy} = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})$ is not necessarily unique while the covariance matrix of $\mathbf{Gy}$ is always unique. The new contributions in Section 3.6 are related to the concept of unit canonical correlations (i.e., those equal to one) between the vector of the ordinary least-squares fitted values $\mathbf{Hy}$ and the vector of the residuals $\mathbf{My}$. This concept, the unit canonical correlations, appears to have had a crucial role also in several other papers later written (or coauthored) by Baksalary; see, e.g., Baksalary, Puntanen and Yanai (1992).

Lemma 2.1 below offers various equivalent characterizations for the situation when there are no unit canonical correlations between $\mathbf{Hy}$ and $\mathbf{My}$; see, e.g., Puntanen (1987, Lemma 4.2.1) and Baksalary, Puntanen and Styan (1990, p. 289). For a brevity, we will denote

$$(2.5) \quad u = \text{ number of unit canonical correlations between } \mathbf{Hy} \text{ and } \mathbf{My}.$$

Lemma 2.1. *Consider the linear model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$. *Then the following eight conditions are equivalent:*

(a) *There are no unit canonical correlations between* $\mathbf{Hy}$ *and* $\mathbf{My}$,

(b) $\mathbf{HP_V} = \mathbf{P_V H}$,

(c) $\mathbf{HP_V M} = \mathbf{0}$,

(d) $\mathscr{C}(\mathbf{VH}) \cap \mathscr{C}(\mathbf{VM}) = \{\mathbf{0}\}$,

(e) $\mathscr{C}(\mathbf{V}^{1/2}\mathbf{H}) \cap \mathscr{C}(\mathbf{V}^{1/2}\mathbf{M}) = \{\mathbf{0}\}$,

(f) $\mathbf{C}(\mathbf{P_V H}) \subset \mathbf{C}(\mathbf{H})$,

(g) $\mathscr{C}(\mathbf{P_V H}) = \mathscr{C}(\mathbf{P_V}) \cap \mathscr{C}(\mathbf{H})$,

(h) $\mathscr{C}(\mathbf{X}) = \mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{V}) + \mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{V})^{\perp}$.

The condition (b) of Lemma 2.1 gives us a good reason to recall that Baksalary did fundamental work on studying the properties of the commuting projectors; see, e.g., Baksalary (1987), where in Theorem 1 he gave 45 equivalent conditions to the commutativity of two orthogonal projectors.

We may cite Puntanen (1987, p. 53) who says that the situations when $\mathbf{V}$ is positive definite or when $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{V})$ are not the only ones yielding "simple" representations for the BLUE's covariance matrix; by "simple" it is here meant representations of the type (2.3) or (2.4). This situation is considered in the following theorem (see also Puntanen and Scott, 1996, Theorem 2.6).

Theorem 2.1. *Consider the linear model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$, and let $\mathbf{X}\tilde{\beta}$ be the* BLUE *of* $\mathbf{X}\beta$*. Then the following six statements are equivalent:*

(a) *There are no unit canonical correlations between* $\mathbf{Hy}$ *and* $\mathbf{My}$,

(b) $\operatorname{cov}(\mathbf{X}\tilde{\beta}) = \mathbf{X}_{\mathrm{o}}(\mathbf{X}_{\mathrm{o}}'\mathbf{V}^{+}\mathbf{X}_{\mathrm{o}})^{+}\mathbf{X}_{\mathrm{o}}'$,

(c) $\operatorname{cov}(\mathbf{X}\tilde{\beta}) = \mathbf{H}(\mathbf{H}\mathbf{V}^{+}\mathbf{H})^{+}\mathbf{H}$,

(d) $\operatorname{cov}(\mathbf{X}\tilde{\beta}) = \mathbf{P}_{\mathbf{V}}\mathbf{H}(\mathbf{H}\mathbf{V}^{+}\mathbf{H})^{-}\mathbf{H}\mathbf{P}_{\mathbf{V}}$,

(e) $\operatorname{cov}(\mathbf{X}\tilde{\beta}) = \mathbf{P}_{\mathbf{V}}\mathbf{X}_{\mathrm{o}}(\mathbf{X}_{\mathrm{o}}'\mathbf{V}^{+}\mathbf{X}_{\mathrm{o}})^{-}\mathbf{X}_{\mathrm{o}}'\mathbf{P}_{\mathbf{V}}$,

(f) $\operatorname{cov}(\mathbf{X}\tilde{\beta}) = \mathbf{P}_{\mathbf{V}}\mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{-}\mathbf{X}'\mathbf{P}_{\mathbf{V}}$,

where $\mathbf{X}_o$ *is a matrix whose columns form an orthonormal basis for* $\mathscr{C}(\mathbf{X})$.

Puntanen (1987, p. 55) mentions that it is somewhat unexpected that in Theorem [2.1][2] we cannot, in general (assuming only $u = 0$), write the equality

$$\mathbf{X}_o(\mathbf{X}_o'\mathbf{V}^{+}\mathbf{X}_o)^{+}\mathbf{X}_o' = \mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{+}\mathbf{X}'. \tag{2.6}$$

Puntanen (coincidentally having a numerical error in his calculations) gives the following counterexample to (2.6):

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{X}_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

[2]If the reference number is between the square brackets, it means that we use the numbering within this paper—not that in the original source.

In this situation $u = 0$ but

$$\mathbf{X}_o(\mathbf{X}_o'\mathbf{V}^+\mathbf{X}_o)^+\mathbf{X}_o' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$
$$= \mathbf{X}(\mathbf{X}'\mathbf{V}^+\mathbf{X})^+\mathbf{X}'.$$

So clearly condition $u = 0$ alone is not enough to guarantee that the covariance matrix of BLUE($\mathbf{X}\boldsymbol{\beta}$) is equal to $\mathbf{X}(\mathbf{X}'\mathbf{V}^+\mathbf{X})^+\mathbf{X}'$. What is needed more is shown in the following result of Puntanen (1987, Theorem 3.6.2):

Theorem 2.2. *Consider the linear model* $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. *Then the following statements are equivalent:*

(a) $\mathrm{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}(\mathbf{X}'\mathbf{V}^+\mathbf{X})^+\mathbf{X}'$,

(b) $u = 0$ *and* $\mathscr{C}(\mathbf{X}'\mathbf{X}\mathbf{X}'\mathbf{V}) = \mathscr{C}(\mathbf{X}'\mathbf{V})$.

Baksalary noted, in private correspondence in summer 1986 – having seen the manuscript of Puntanen's thesis – that Theorems 2.1 and 2.2 are related to the Theorem in Baksalary and Kala (1980, p. 19). That theorem concerns the following estimator suggested by Ahlers and Lewis (1971):

$$\mathbf{X}(\mathbf{X}'\mathbf{V}^+\mathbf{X})^+\mathbf{X}'\mathbf{V}^+\mathbf{y} + \mathbf{X}(\mathbf{X}'\mathbf{Q}_\mathbf{V}\mathbf{X})^+\mathbf{X}'\mathbf{Q}_\mathbf{V}\mathbf{y} := \mathbf{A}\mathbf{y}, \tag{2.9}$$

where $\mathbf{Q}_\mathbf{V} = \mathbf{I} - \mathbf{P}_\mathbf{V}$.

Baksalary and Kala (1980) proved that the estimator (2.9) is the BLUE if and only if

$$\mathscr{C}(\mathbf{X}\mathbf{X}'\mathbf{V}) \subset \mathscr{C}(\mathbf{V}). \tag{2.10}$$

Noting that

$$\mathrm{cov}(\mathbf{Ay}) = \mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{+}\mathbf{X}', \tag{2.11}$$

we can conclude that condition (2.10) implies that BLUE's covariance matrix has a representation $\mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{+}\mathbf{X}'$. We note that the Ahlers and Lewis' estimator was also studied by Alalouf (1975a, p. 182; 1975b, p. 101), who gave the condition (2.10) in an alternative form.

Based on the above findings, Puntanen (1987, Theorem 3.6.4) combined the results as follows:

Theorem 2.3. *Consider the linear model* $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. *Then the following four statements are equivalent:*

(a) $\mathrm{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{+}\mathbf{X}'$,

(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}^{+}\mathbf{y} + \mathbf{X}(\mathbf{X}'\mathbf{Q}_{\mathbf{V}}\mathbf{X})^{+}\mathbf{X}'\mathbf{Q}_{\mathbf{V}}\mathbf{y}$,

(c) $u = 0$ *and* $\mathscr{C}(\mathbf{X}'\mathbf{X}\mathbf{X}'\mathbf{V}) = \mathscr{C}(\mathbf{X}'\mathbf{V})$,

(d) $\mathscr{C}(\mathbf{X}\mathbf{X}'\mathbf{V}) \subset \mathscr{C}(\mathbf{V})$.

2.2. Developments after 1987

How did the things develop after Puntanen's thesis?

In the thesis (p. 51) Puntanen writes that "a direct proof that [(2.1b)] and [(2.1d)] are equal is given in Puntanen and Styan (1986)" That paper was a manuscript under preparation which was submitted to *Sankhyā* in July 1987. We (Puntanen and Styan) gave a copy to Baksalary and on 5 November 1987 he wrote us a letter starting as follows:

> *Dear George and Simo:*
>
> *In some "spare time", I have had a look at your paper "More properties of the covariance matrix of the BLUE in the general linear model" enclosed to Simo's letter of July 16, 1987, and now I am taking a liberty of making some comments on it. Perhaps you would find them useful in further processing with that paper.*

The referee reports from *Sankhyā* arrived in January 1988. The reports were exceptionally constructive and carefully done, requesting, however, a revision. [We believe that one of the two referees was the late Professor C. G. Khatri since he published a related paper in 1990 in *Journal of Multivariate Analysis*, where he referred to our manuscript.]

At this phase, in spring 1988, we invited Baksalary into the game, and that was very wisely done: this was just the kind of research area where he was a master. Under the leadership of Baksalary, the manuscript was thoroughly revised and all results were generalized in the Baksalarian style to the maximum.

Baksalary completed the revision while he was visiting Professor C. Radhakrishna Rao in Pittsburgh, for four weeks in summer 1988. The paper was submitted to *Sankhyā* in July 1988, and it was published in 1990.

Below we copy a letter, dated 8 July 1988, from Baksalary to us. We believe that it gives an interesting illustration of Baksalary's working style.

Pittsburgh, 8 July 1988

Dear Simo:

I hope you have already received my comments on BPS (version of 3 July 1988); these comments are seen on the present copy in black. New changes are in red. Sorry, but I was unable to go again through Sections 3 and 4. Notice that the version of Section 2 begun with discovering an error. Nevertheless, it seems that the paper is now more or less ready to be handled. When I get the typed version from Tampere (please don't staple), I put it immediately into Technical Report Series here. The next proofreading in Pittsburgh is impossible because of time factor. Simultaneously, you may submit the paper to *Sankhyā*. I suggest no long story in the covering letter, just the statement that, due to remarks of referees, we were able to produce a much more general, substantially better paper which is now being submitted.

Parenthetically, I may mention that the open problem raised in Section 2 is no longer open. Thomas Mathew visited Pittsburgh for 4 days and we solved it in the most general form concerning the invariance of $\mathrm{r}(\mathbf{AB}^-\mathbf{C})$. We will send you a copy, of course, when we get the paper typed. It will also be sent to George as an editor of SILAX.

Hoping that we will be able to finish successfully the BPS1 adventure,

Jerzy

We may write here, to illustrate Baksalary's capacity to generalize previous results, two theorems (Corollary 2 and Theorem 5) from Baksalary, Puntanen and Styan (1990):

Theorem 2.4. *Let* $\mathbf{C}$ *be an* $n \times n$ *matrix, let* $\mathbf{A}$ *be an* $n \times p$ *matrix of rank* a. *Then, for any* $n \times a$ *matrix* $\mathbf{A}_*$ *such that* $\mathscr{C}(\mathbf{A}_*) = \mathscr{C}(\mathbf{A})$ *and* $\mathbf{A}_*'\mathbf{A}_* = \mathbf{I}_a$, *the equality*

$$\mathbf{A}_*(\mathbf{A}_*'\mathbf{C}\mathbf{A}_*)^+\mathbf{A}_*' = \mathbf{P}_\mathbf{A}(\mathbf{P}_\mathbf{A}\mathbf{C}\mathbf{P}_\mathbf{A})^+\mathbf{P}_\mathbf{A} = (\mathbf{P}_\mathbf{A}\mathbf{C}\mathbf{P}_\mathbf{A})^+ \tag{2.12}$$

is always true, whereas the equality

$$\mathbf{A}_*(\mathbf{A}_*'\mathbf{C}\mathbf{A}_*)^+\mathbf{A}_*' = \mathbf{A}(\mathbf{A}'\mathbf{C}\mathbf{A})^+\mathbf{A}' \tag{2.13}$$

holds if and only if

$$\mathbf{C}(\mathbf{A}'\mathbf{A}\mathbf{A}\mathbf{C}\mathbf{A}) = \mathbf{C}(\mathbf{A}'\mathbf{C}\mathbf{A}). \tag{2.14}$$

Theorem 2.5. *Consider the linear model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *and let* $\mathbf{X}_*$ *be a given matrix such that* $\mathscr{C}(\mathbf{X}_*) = \mathscr{C}(\mathbf{X})$. *Then the following five statements are equivalent:*

(a) $\mathrm{cov}[\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathbf{X}_*(\mathbf{X}_*'\mathbf{V}^+\mathbf{X}_*)^+\mathbf{X}_*'$,

(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}_*(\mathbf{X}_*'\mathbf{V}^+\mathbf{X}_*)^+\mathbf{X}_*'\mathbf{V}^+\mathbf{y} + \mathbf{X}_*(\mathbf{X}_*'\mathbf{Q}_\mathbf{V}\mathbf{X}_*)^+\mathbf{X}_*'\mathbf{Q}_\mathbf{V}\mathbf{y}$,

(c) $u = 0$ *and, moreover,* $\mathscr{C}(\mathbf{X}_*'\mathbf{X}_*\mathbf{X}_*'\mathbf{V}) = \mathscr{C}(\mathbf{X}_*'\mathbf{V})$ *or* $\mathscr{C}(\mathbf{X}_*'\mathbf{X}_*\mathbf{F}) = \mathscr{C}(\mathbf{F})$, *where* $\mathbf{F}$ *is any matrix such that* $\mathscr{C}(\mathbf{F}) = \mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{V})$,

(d) $\mathscr{C}(\mathbf{X}_*\mathbf{X}_*'\mathbf{V}) \subset \mathscr{C}(\mathbf{V})$,

(e) $\mathbf{X}_*\mathbf{X}_*'\mathbf{P}_\mathbf{V} = \mathbf{P}_\mathbf{V}\mathbf{X}_*\mathbf{X}_*'$.

3. THE MATRIX $\dot{\mathbf{M}}$

In this section we briefly consider the matrix

$$\dot{\mathbf{M}} = \mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}, \tag{3.1}$$

which appeared already in (2.2). We observe that the matrix $\dot{\mathbf{M}}$ is not necessarily unique with respect to the choice of $(\mathbf{MVM})^{-}$. However, the matrix product

$$\mathbf{P_V}\dot{\mathbf{M}}\mathbf{P_V} = \mathbf{P_V}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}\mathbf{P_V} := \ddot{\mathbf{M}} \tag{3.2}$$

is clearly invariant for any choice of $(\mathbf{MVM})^{-}$, i.e.,

$$\ddot{\mathbf{M}} = \mathbf{P_V}\dot{\mathbf{M}}\mathbf{P_V} = \mathbf{P_V}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}\mathbf{P_V} = \mathbf{P_V}\mathbf{M}(\mathbf{MVM})^{+}\mathbf{M}\mathbf{P_V}. \tag{3.3}$$

The matrices $\dot{\mathbf{M}}$ and $\ddot{\mathbf{M}}$ appear to be very handy in many ways related to linear model $\mathcal{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}\}$. In the recent paper Isotalo, Puntanen and Styan (2008), we collect together various properties of $\dot{\mathbf{M}}$ and $\ddot{\mathbf{M}}$ and show several examples illustrating their usefulness in the context of linear models.

Below is the abstract of this paper:

> It is well known that if $\mathbf{V}$ is a symmetric positive definite $n \times n$ matrix, and $(\mathbf{X} : \mathbf{Z})$ is a partitioned orthogonal $n \times n$ matrix, then
>
> $$(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} = \mathbf{X}'\mathbf{V}\mathbf{X} - \mathbf{X}'\mathbf{V}\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{V}\mathbf{X}. \tag{*}$$
>
> In this paper we show how useful we have found the formula $(*)$, and in particular its version
>
> $$\mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^{-1}\mathbf{Z}' = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1} := \dot{\mathbf{M}}, \tag{**}$$
>
> and present several related formulas, as well as some generalized versions. We also include several statistical applications.

Note that if $\mathbf{V}^{1/2}$ is a positive definite symmetric square root of $\mathbf{V}$, and the columns of $\mathbf{Z}$ are spanning $\mathscr{C}(\mathbf{M})$, then we obviously have

$$\begin{aligned} \mathbf{Z}(\mathbf{Z}'\mathbf{V}\mathbf{Z})^{-}\mathbf{Z}' &= \mathbf{V}^{-1/2}\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{Z}}\mathbf{V}^{-1/2} = \mathbf{V}^{-1/2}\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{M}}\mathbf{V}^{-1/2} \\ &= \mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{-}\mathbf{M}. \end{aligned} \tag{3.4}$$

The following theorem characterizes some properties of $\dot{\mathbf{M}}$ and $\ddot{\mathbf{M}}$; for a more complete list, see Isotalo, Puntanen and Styan (2008, Theorem 2.1).

Theorem 3.1. *Consider the linear model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, *and let the matrices* $\mathbf{M}$, $\dot{\mathbf{M}}$, *and* $\ddot{\mathbf{M}}$ *be defined as*

$$\mathbf{M} = \mathbf{I} - \mathbf{P}_{\mathbf{X}}, \qquad \dot{\mathbf{M}} = \mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{-}\mathbf{M}, \qquad \ddot{\mathbf{M}} = \mathbf{P}_{\mathbf{V}}\dot{\mathbf{M}}\mathbf{P}_{\mathbf{V}}. \tag{3.5}$$

Assume that the condition

$$\mathbf{H}\mathbf{P}_{\mathbf{V}}\mathbf{M} = \mathbf{0} \tag{3.6}$$

holds. Then

$$\ddot{\mathbf{M}} = \mathbf{P}_{\mathbf{V}}\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{-}\mathbf{M}\mathbf{P}_{\mathbf{V}} = \mathbf{V}^{+} - \mathbf{V}^{+}\mathbf{X}(\mathbf{X}'\mathbf{V}^{+}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{+}. \tag{3.7}$$

The point here is that Theorem 3.1 can be seen as a consequence of the *Sankhyā* paper by Baksalary, Puntanen and Styan (1990): the various representations of the BLUE's covariance matrix yielded the appearence of $\dot{\mathbf{M}}$ and $\ddot{\mathbf{M}}$. We met these matrices in many statistical connections and that motivated us to write the paper Isotalo, Puntanen and Styan (2008).

Before proceeding onwards, we present the following useful lemma.

Lemma 3.1. *Consider the linear model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *and let* $\mathbf{W} = \mathbf{V} + \mathbf{X}\mathbf{U}\mathbf{X}'$, *where* $\mathbf{U}$ *is a* $p \times p$ *matrix. Then the following seven statements are equivalent:*

(a) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W})$,

(b) $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$,

(c) $\operatorname{rank}(\mathbf{X} : \mathbf{V}) = \operatorname{rank}(\mathbf{W})$,

(d) $\mathbf{X}'\mathbf{W}^{-}\mathbf{X}$ *is invariant for any choice of* $\mathbf{W}^{-}$,

(e) $\mathscr{C}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})$ *is invariant for any choice of* $\mathbf{W}^{-}$,

(f) $\mathscr{C}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X}) = \mathscr{C}(\mathbf{X}')$ *for any choice of* $\mathbf{W}^{-}$,

(g) $\operatorname{rank}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X}) = \operatorname{rank}(\mathbf{X})$ *irrespective of the choice of* $\mathbf{W}^{-}$.

Moreover, each of these statements is equivalent to

(a′) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}')$,

and hence equivalent to the statements (b′)–(g′) *obtained from* (b)–(g), *by substituting* $\mathbf{W}'$ *for* $\mathbf{W}$.

The proof of Lemma 3.1 is given by Baksalary, Puntanen and Styan (1990, Theorem 2) (see also Harville, 1997, p. 468). On page 284, Baksalary, Puntanen and Styan (1990) state the following (in our notation):

> ...Here we only mention that it would be interesting to know whether the statements of [Lemma 3.1] are equivalent also to the rank condition relaxed to the requirement that $\mathrm{r}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})$ is invariant with respect to the choice of $\mathbf{W}^{-}$.

This was a reference to the letter from Baksalary (8 July 1988) from Pittsburgh cited earlier, where " ...the open problem raised in Section 2 is no longer open ...". The solution was published in a paper by Baksalary and Mathew[3] (1990, Theorem 2) and it stated that the following condition can be added into the set of equivalent conditions of Lemma 3.1:

(3.8) $\operatorname{rank}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})$ is invariant for any choice of $\mathbf{W}^{-}$.

[3]As a curiosity, we may mention that Jerzy K. Baksalary was a referee of Thomas Mathew's Ph.D. dissertation in 1983, while Thomas Mathew was a referee of Jarkko Isotalo's Ph.D. dissertation just recently in 2007.

Next we present a generalized version of Theorem 3.1 as a corollary; see Isotalo, Puntanen and Styan (2008, Corollary 2.2).

Corollary 3.1. *Consider the linear model $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. Let $\mathbf{U}$ be any $p \times p$ matrix such that the matrix $\mathbf{W} = \mathbf{V} + \mathbf{XUX}'$ satisfies the condition*

$$\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}). \tag{3.9}$$

Then

$$\mathbf{P_W M(MWM)^- MP_W} = \mathbf{W}^+ - \mathbf{W}^+\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+, \tag{3.10}$$

that is,

$$\mathbf{P_W \dot{M} P_W} := \ddot{\mathbf{M}}_{\mathrm{W}} = \mathbf{W}^+ - \mathbf{W}^+\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+. \tag{3.11}$$

Moreover, the matrix $\ddot{\mathbf{M}}_{\mathrm{W}}$ has the corresponding properties as $\ddot{\mathbf{M}}$ in Theorem 3.1.

We complete this section with a generalization of the decomposition presented in Corollary 3.1. It seems to us that this formulation, due to Baksalary, Puntanen and Styan (1990, Theorem 3), is one of the most general formulations related to $\dot{\mathbf{M}}$. In fact, it may even be "too general" in the sense that statisticians may overlook it in favour of the possibly more "useful" decomposition in Corollary 3.1.

Theorem 3.2. *Consider the model $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ and let $\mathbf{U}$ be such that $\mathbf{W} = \mathbf{V} + \mathbf{XUX}'$ satifies $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$. Then the equality*

$$\mathbf{W} = \mathbf{VB}(\mathbf{B}'\mathbf{VB})^-\mathbf{B}'\mathbf{V} + \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}' \tag{3.12}$$

holds for an $n \times p$ matrix $\mathbf{B}$ if and only if

$$\mathscr{C}(\mathbf{VW}^-\mathbf{X}) \subset \mathscr{C}(\mathbf{B})^\perp \text{ and } \mathscr{C}(\mathbf{VM}) \subset \mathscr{C}(\mathbf{VB}), \tag{3.13}$$

or, equivalently,

$$\mathscr{C}(\mathbf{VW}^-\mathbf{X}) = \mathscr{C}(\mathbf{B})^\perp \cap \mathscr{C}(\mathbf{V}), \tag{3.14}$$

the subspace $\mathscr{C}(\mathbf{VW}^-\mathbf{X})$ being independent of the choice of $\mathbf{W}^-$.

4. LINEAR SUFFICIENCY AND COMPLETENESS

Linear sufficiency and linear completeness are one of the most important concepts of Isotalo's thesis. Baksalary did substantial work on this area, and hence we here briefly review his work on linear sufficiency and completeness and show its connections to Isotalo's thesis; see, in particular Isotalo and Puntanen (2006a, 2006b, 2006c).

The concept of linear sufficiency was introduced by Barnard (1963), Baksalary and Kala (1981), and Drygas (1983)—who was the first to use the term linear sufficiency—while investigating those linear statistics $\mathbf{Ty}$, which are "sufficient" for estimation of the expected value $\mathbf{X}\boldsymbol{\beta}$ in the model $\mathcal{M}$. Formally, a linear statistic $\mathbf{Ty}$ is defined to be linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under the model $\mathcal{M}$ if there exists a matrix $\mathbf{A}$ such that $\mathbf{ATy}$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$. Baksalary and Kala (1981, p. 913) illustrate the situation in the following "concrete" way (in our notation):

> If the vector $\mathbf{y}$ subject to the model $\{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{I}\}$ were transformed into the $\mathbf{w} = \mathbf{X}'\mathbf{y}$, then the BLUE of $\mathbf{X}\boldsymbol{\beta}$,
>
> $$\mathbf{X}\tilde{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y},$$
>
> would be obtainable as a linear function of $\mathbf{w}$, namely as
>
> $$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{w}.$$
>
> If, however, the same transformation were adopted under the model $\{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}\}$ ($\mathbf{V}$ positive define but different from $\mathbf{I}$) then the BLUE of $\mathbf{X}\boldsymbol{\beta}$, having now the form
>
> $$\mathbf{X}\tilde{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y},$$
>
> would no longer be obtainable as a linear function of $\mathbf{w} = \mathbf{X}'\mathbf{y}$ unless $\mathscr{C}(\mathbf{V}^{-1}\mathbf{X}) \subset \mathscr{C}(\mathbf{X})$. This exception might in fact be expected as the inclusion is a necessary and sufficient condition for the OLSE and BLUE to be identical (Haberman, 1975).

Drygas (1983, p. 97) points out, in his section entitled "Historical remarks": "The concept of linearly sufficient statistics is rather unknown in statistical literature. Besides the paper by Baksalary and Kala (1978) [this refers to the paper later published in 1981] there is only one paper by Barnard (1963) (see also Cox and Hinkley, 1974, p. 61)."–

Apparently those days Drygas, Baksalary and Kala were pretty well aware of each others' doings related to the linear sufficiency.

Baksalary and Kala (1981) and Drygas (1983) showed that a linear statistic $\mathbf{Ty}$ is linearly sufficient for $\mathbf{X}\boldsymbol{\beta}$ under the model $\mathcal{M}$ if and only if the column space inclusion

$$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{WT}') \tag{4.1}$$

holds; here $\mathbf{W} = \mathbf{V} + \mathbf{XUX}'$ with $\mathbf{U}$ being an arbitrary nonnegative definite matrix such that $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$.

In addition to linear sufficiency, Drygas (1983) also considered related concepts of linear minimal sufficiency and linear completeness. A linearly sufficient statistic $\mathbf{Ty}$ is called linearly minimal sufficient for $\mathbf{X}\boldsymbol{\beta}$ under the model $\mathcal{M}$, if for any other linearly sufficient statistic $\mathbf{Sy}$, there exists a matrix $\mathbf{A}$ such that $\mathbf{Ty} = \mathbf{ASy}$ almost surely. Drygas (1983) showed that $\mathbf{Ty}$ is linearly minimal sufficient for $\mathbf{X}\boldsymbol{\beta}$ if and only if the equality

$$\mathscr{C}(\mathbf{X}) = \mathscr{C}(\mathbf{WT}') \tag{4.2}$$

holds.

Moreover, Drygas (1983) called a linear statistic $\mathbf{Ty}$ linearly complete if for every linear transformation of it, $\mathbf{LTy}$, such that $\mathrm{E}(\mathbf{LTy}) = \mathbf{0}$, it follows that $\mathbf{LTy} = \mathbf{0}$ almost surely. According to Drygas (1983), a linear statistic $\mathbf{Ty}$ is linearly complete if and only if

$$\mathscr{C}(\mathbf{TV}) \subset \mathscr{C}(\mathbf{TX}). \tag{4.3}$$

It was then shown by Drygas (1983) that a linear statistic $\mathbf{Ty}$ is linearly minimal sufficient for $\mathbf{X}\boldsymbol{\beta}$ if and only if it is simultaneously linearly sufficient and linearly complete for $\mathbf{X}\boldsymbol{\beta}$.

Baksalary and Kala (1986) extended the notions of linear sufficiency and linear minimal sufficiency to concern estimation of the given estimable parametric function $\mathbf{K}'\boldsymbol{\beta}$. They proved that $\mathbf{Ty}$ is linearly sufficient for $\mathbf{K}'\boldsymbol{\beta}$ under the model $\mathcal{M}$ if and only if the null space inclusion

$$\mathscr{N}(\mathbf{TX} : \mathbf{TVX}^{\perp}) \subset \mathscr{N}(\mathbf{K}' : \mathbf{0}) \tag{4.4}$$

holds, and $\mathbf{Ty}$ is linearly minimal sufficient for $\mathbf{K}'\boldsymbol{\beta}$ if and only if the null space equality

$$\mathscr{N}(\mathbf{TX} : \mathbf{TVX}^{\perp}) = \mathscr{N}(\mathbf{K}' : \mathbf{0}) \tag{4.5}$$

holds.

However in their paper, Baksalary and Kala (1986) did not consider linear completeness in the context of estimating the given estimable parametric function $\mathbf{K}'\boldsymbol{\beta}$. But Baksalary clearly had an interest to generalize the concept of linear completeness also to the case of estimation of $\mathbf{K}'\boldsymbol{\beta}$. Baksalary had previously in his Habilitation Thesis (1984) considered linear completeness under the more simple model $\{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \sigma^2\mathbf{I}\}$. Baksalary (1984) defines a linear statistic $\mathbf{Ty}$ to be linearly complete for $\mathbf{K}'\boldsymbol{\beta}$ if for every linear transformation of it, $\mathbf{LTy}$, such that $\mathscr{C}(\mathbf{T}'\mathbf{L}') \subset \mathscr{C}[(\mathbf{X}')^{+}\mathbf{K}]^{\perp}$, it follows that $\mathbf{LTy} = \mathbf{0}$ almost surely.

Baksalary's insights on linear completeness were then source of inspiration to the results on completeness presented in Isotalo's dissertation. In the second article of Isotalo's thesis, Isotalo and Puntanen (2006b) give a definition for linear completeness in a case of estimation of $\mathbf{K}'\boldsymbol{\beta}$ which has same implications as Baksalary's (1984) definition but has more resemblance to the original definition given by Drygas (1983). Isotalo and Puntanen's (2006b) definition of linear completeness is based on the following reparametrized model of $\mathcal{M}$:

$$\begin{aligned}\mathcal{M}_{\gamma} &= \{\mathbf{y},\, \mathbf{X}(\mathbf{K} : \mathbf{K}^{\perp})\boldsymbol{\gamma},\, \sigma^2\mathbf{V}\} \\ &= \{\mathbf{y},\, \mathbf{XK}\boldsymbol{\gamma}_1 + \mathbf{XK}^{\perp}\boldsymbol{\gamma}_2,\, \sigma^2\mathbf{V}\},\end{aligned} \tag{4.6}$$

where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2')'$. In their article (2006b), Isotalo and Puntanen first prove that the BLUE of $\mathbf{K}'\boldsymbol{\beta}$ under the original model $\mathcal{M}$ is equivalent to the BLUE of $\mathbf{K}'\mathbf{K}\boldsymbol{\gamma}_1$ under the reparametrized model $\mathcal{M}_{\gamma}$, and then define a linear statistic $\mathbf{Ty}$ to be linearly complete for $\mathbf{K}'\boldsymbol{\beta}$ if for every linear transformation of it, $\mathbf{LTy}$, such that the expected value $\mathrm{E}(\mathbf{LTy})$ does not dependent on $\boldsymbol{\gamma}_1$ under the reparametrized model $\mathcal{M}_{\gamma}$, it follows that $\mathbf{LTy} = \mathbf{0}$ almost surely.

Now by using their definition of linear completeness, Isotalo and Puntanen (2006b) were then able to prove an important property of a linear statistic $\mathbf{Ty}$ being linearly minimal sufficient for $\mathbf{K}'\boldsymbol{\beta}$ if and only if it is simultaneously linearly sufficient and linearly complete for $\mathbf{K}'\boldsymbol{\beta}$.

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