

Lectures on
Matrix and Graph Methods

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Foreword



Department of Statistics of Manipal University successfully conducted an International Workshop and Conference on Combinatorial Matrix Theory & Generalized Inverses of Matrices in January, 2012. This well attended academic event had lectures from the likes of Steve Kirkland, Sukanta Pati, Sivaramakrishnan, Jeffrey Hunter, Simo Puntanen, Balakrishnan, Rajendra Bhatia, T.E.S. Raghavan, George Styan, Oskar Baksalary, Götz Trenkler etc.

Manipal University Press (MUP) felt that it was necessary to disseminate these lectures worldwide and undertook to publish the lecture notes and also digitize the contents for electronic transmission. MUP has been spearheading the cause of enhancing the academic throughput of the University and also make these works available to Universities, academicians and libraries across the world. The lecture notes on Linear Algebra opens up a new genre of academic publication.

Manipal Centre for Natural Sciences (MCNS) will soon be establishing a division of Mathematics & Computational Sciences. This will be one of many divisions that MCNS will initiate and incubate.

Dr. Manjunatha Prasad, Dr. Sreekumaran Nair and faculty of Department of Statistics have strived hard to make a success of the workshop and the publication of this lecture notes is a testimony to that effort.

Thanks to all the authors for their excellent contributions and to the editorial team lead by Dr. R.B. Bapat for their effort in bringing out this volume.

May 14, 2012

Dr. H. Vinod Bhat
Pro Vice-Chancellor, Manipal University

Preface

Generalized inverses of matrices and combinatorial matrix theory are among the areas of matrix theory with a strong theoretical component as well as with applications in diverse areas that have seen rapid advances in recent years. Significant contribution to generalized inverses has come from the Indian school including luminaries such as C.R. Rao, S.K. Mitra and C.G. Khatri. Incidentally, the eightieth birthday of Professor S.K. Mitra was on January 23, 2012, in the same month that the “International workshop on combinatorial matrix theory and generalized inverse of matrices” was held.

The interaction between graph theory and matrix theory is known to be very fruitful and examples abound where generalized inverses of matrices arising from graphs play an important role. Just to give one example, an expression for the resistance distance between two vertices in a graph in terms of a generalized inverse of the Laplacian matrix is a key tool.

The six-day workshop provided an ideal opportunity for participants and speakers from India and abroad to interact and exchange ideas in pleasant surroundings. For the benefit of the participants, for those who were interested but failed to participate due to unavoidable circumstances, and for the general interested reader, the organizers planned to bring out this contributed volume covering selected lectures delivered in the workshop. The volume will be distributed to all the participants free of cost. Thus, the volume can fruitfully be used by the participants of the workshop and hopefully by others as well, as a valuable resource material. These notes cover the contents of the actual lectures delivered in the limited hours during the workshop, in greater depth and proper organization. Hence, this volume should be a rich resource of knowledge in its respective areas.

The organizers received overwhelming response from almost all the speakers who contributed their chapters based on their lectures. The editors wish to thank them and their collaborators.

For the benefit of young scholars, some introductory topics are included. Professor Sukanta Pati introduces matrix-graph relations and algebraic connectivity of graphs, whereas Professor K.M. Prasad introduces generalized inverse of matrices and their algebraic structure.

Professor Jeffrey Hunter, in his chapter, introduces generalized inverse and elaborates on its applications to many problems related to Markov chains such as stationary distributions, moments of first passage time distributions and moments of occupation time random variables, and to the problems concerned with the solution of a system of linear equations involving $I - P$, where P is the transition matrix of a finite, irreducible and discrete time Markov chain.

Professor R. Balakrishnan, in his chapter, discusses the spectral properties of graphs and deals with the energy of a graph, a graph parameter that has close links to chemistry and Ramanujan graphs. Professor

Sivaramakrishnan introduces \mathcal{L}_q , the q -analogue of the Laplacian of a graph G and explores several properties of this matrix.

Professor Steve Kirkland, in his chapter, introduces spectral graph theory and discusses graph structure revealed by the spectral properties of the Laplacian matrix. He elaborates on a number of results that reveal the connection between the eigenvalues and the connectivity properties of graphs. The smallest eigenvalue of the signless Laplacian matrix and its role as a measure of bipartiteness of the graph is discussed as well.

In matrix algebra, there are handy, sometimes even very simple “tricks” which simplify and clarify the treatment of a problem (in the area of linear statistical models) both for the student and the teacher. Professor S. Puntanen, in his lecture in the workshop, has shared a few matrix ‘TRICKS’ from his collection. The same is presented as a chapter in the present volume. Professors Oskar Maria Baksalary and Götz Trenkler, in the workshop, have shared their knowledge concerning projectors and their applications, which they gained through their journey of research during last few decades.

Professor R. Bhatia in his lecture at the workshop elaborated upon a few general principles like X -ification, Berberian tricks and unitary invariance, which have been found useful in deriving some important matrix inequalities such as the arithmetic-geometric mean inequality and the Cauchy-Schwarz inequality, where the matrices involved are positive semidefinite. Continuing further, applications of the general principles in obtaining perturbation bounds were discussed. K.S. Mohana and Y. Santhi Sheela have compiled his lecture notes and presented the same in this volume.

Professor H.J. Werner in his lecture at the workshop, introduced g -inverses, projectors and their utility in the study of general Gauß-Markov model.

Lectures in the workshop covered the topics even beyond the traditional applications of matrix theory and spectral theory of graphs. Lectures by Professors T.E.S. Raghavan, George P.H. Styan and S. Ganesan are such ones. Professor Raghavan introduced ‘Cooperative Transfer Utility’ game and discussed two fundamental solution concepts – the *Shapley value* and the *nucleolus* for cooperative TU games. An efficient algorithm he discussed was based on some judicious applications of graph theoretic tools. Professor S. Ganesan discussed the importance of covariance matrix in handling the nuclear data. Professor Styan gave a lecture over the Skype which was well received by the participants. He introduced Agrippa type magic matrices and algorithms to generate such magic matrices. He further elaborated on magic knight’s tours and introduced the concepts of EP-, bi-EP-, and bP-multipliers to measure how far a magic matrix is from being, respectively, EP, bi-EP or bP (bi-EP but not EP).

Lectures by Werner and Ganesan are not included in this volume.

This volume also presents the contents of an interview with a great mathematician who inspired every one working in the area of matrix methods and generalized inverse. Professors Styan and Puntanen had interviewed Professor Sujit Kumar Mitra in February, 1993, and the same is given here as inspiration for young mathematicians.

The editors take this opportunity to thank all the speakers who contributed the contents to this volume and also acknowledge their gratitude to ‘Manipal University’ for organizing the workshop.

May 14, 2012

Ravindra B. Bapat
Steve Kirkland
K. Manjunatha Prasad
Simo Puntanen

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The Editors are thankful to Dr. H. Vinod Bhat, Pro Vice-Chancellor, Manipal University and the Chairman of the Organizing Committee of CMTGIM workshop, for being instrumental in the successful conduct of this workshop.

We further thank, Prof. N. Sreekumaran Nair, Head, Department of Statistics, Manipal University for his advice and active support. The following colleagues and research students have taken active part in preparing the notes, putting them in \LaTeX and proof reading this volume: Dr. G. Sudhakara - Professor, Mr. K.S. Mohana - Assistant Professor, Mr. Vinay Madhusudanan - Assistant Professor, Ms. Anitha Raghunathan - Assistant Professor from the Department of Mathematics, MIT Manipal; Dr. Sreemathi S. Mayya - Associate Professor, Ms. Ashma Monterio - Assistant Professor, Ms. Santhi - Research Scholar from the Department of Statistics, Manipal University. The Editors proudly applaud their effort.

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May 14, 2012

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