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SOME COMMENTS ON PHILATELIC LATIN SQUARES FROM PAKISTAN[1]

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ABSTRACT

We explore the use of Latin squares in printing postage stamps, with special emphasis on stamps from Pakistan. We note that Pakistan may be the only country to have issued postage stamps in 2×2, 3×3, 4×4 and 5×5 Latin square formats: we call such sets of stamps *philatelic Latin squares* (PLS). We find that a 5×5 PLS from Pakistan has the knight's move or Knut Vik design as discussed, e.g., by Nissen [56] and Tedin [78], and it seems to be the only 5×5 PLS (from any country) with this property. We identify a 6×5 sheet of stamps from Pakistan depicting 10 kinds of mushrooms (each repeated 3 times) and study two associated 5×5 Latin squares; from these findings we construct a 10×10 "philatelic Sudoku puzzle" and give its solution. An annotated set of over 90 references, many with hyperlinks, ends the paper.

KEYWORDS

Asian Advertising Congress, Mustafa Kemal Atatürk (1881–1938), Walter-Ulrich Behrens (1902–1962), block-Latin rank formula, Raj Chandra Bose (1901–1987), brickwall 10×10 Sudoku format, circulants, composite Latin square, experimental design, Sir Ronald Aylmer Fisher (1890–1962), gerechte Latin squares, Quaid-e-Azam Muhammad Ali Jinnah (1876–1948), knight's move design, Knut Vik design, Kronecker sum, Latin rectangle, Maack: Sudoku-download, mushrooms, pandiagonal Latin square, philatelic Sudoku puzzle, philatelic Youden puzzle, postage stamps, rainbow-Latin square, rank of Latin square matrix, Sudoku, ThinkFun: Everybody Plays, tropical fish, water birds, Frank Yates (1902–1994), William John Youden (1900–1971), Youden square.

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1 INTRODUCTION AND PHILATELIC LATIN SQUARES OF SIZE 2×2

We define a *Latin square* of size $k \times k$ as a square array of k symbols in k rows and k columns such that every symbol occurs once in each row and once in each column. It seems that Latin squares have been so named since Euler [35] used Latin letters for the symbols [83, p. 22]. There is a huge literature on Latin squares, see, e.g., [2, 3, 4, 10, 17, 21, 22, 30, 31, 32, 41, 42, 43, 52, 58, 64, 66, 74, 75, 77]. It was observed by Ritter [66] that:

> It seems that Latin squares were originally mathematical curiosities, but serious statistical applications were found early in the 20th century, as experimental designs. The classic example is the use of a Latin square configuration to place 3 or 4 different grain varieties in test patches. Having multiple patches for each variety helps to minimize localized soil effects. Similar statements can be made about medical treatments.

In their book *Discrete Mathematics Using Latin Squares*, Laywine & Mullen [52, ch. 12, p. 188] noted

> While the study of Latin squares is usually traced to a famous problem involving 36 officers considered in 1779 [34] by Leonhard Euler (1707–1783) [see our Section 4 below], statistics provided a major motivation for important work in the early and middle decades of the 20th century. Indeed many of the most important combinatorial results pertaining to Latin squares were obtained by researchers whose primary self-description would be that of a statistician.

In the Second Edition of their *Handbook of Combinatorial Designs*, Colbourn & Dinitz [21, p. 21] identify, in their timeline of the main contributors to design theory from antiquity to 1950 (consecutively, born between 1890 and 1902):

Sir Ronald Aylmer Fisher (1890–1962),
William John Youden (1900–1971),
Raj Chandra Bose (1901–1987),
Frank Yates (1902–1994).

Latin squares have, however, been used in the statistical design of experiments for over 200 years. As described by Preece in his excellent survey on "R. A. Fisher and experimental design: a review" [62], probably the earliest use of a Latin square in an experimental design was by the agronomist François Cretté de Palluel (1741–1798), who in 1788 published a study [25,26, 73] of an experiment involving the fattening of 16 sheep in France, four of each of four different breeds. The advantage of using a Latin-square design in this experiment was that only 16 sheep were needed rather than 64 for a completely cross-classified design. The purpose here was to show that one might just as well feed sheep on root vegetables during the winter; this was much cheaper, and easier, than the normal diet of corn and hay.

In this paper we explore the use of Latin squares in printing postage stamps with special emphasis on stamps from Pakistan. Postage stamps are occasionally issued in sets of k different stamps and printed in a $k \times k$ array of k^2 stamps containing k of each of the k stamps [53]. Sometimes the array forms a *philatelic Latin square:* each of the k stamps appears exactly once in each row and once in each column; to illustrate, in Figure 1.1, with $k = 2$, with stamps issued by Pakistan in 2005 for the 85th Anniversary of the Turkish Grand National Assembly (see also Figure 1.2). We will use the initialism PLS to denote either "philatelic Latin square" or its plural.

FIGURE 1.1: Top-left 2×2 corner of the 5×4 sheetlet (Figure 1.2) issued by Pakistan 2005, *Scott*[2] 1063a–1063b.

[2]Scott catalog numbers are as given in the *Scott Standard Postage Stamp Catalogue* [47].

The Nobel laureate Ernest Rutherford, 1st Baron Rutherford of Nelson (1871–1937) said: “All science is either physics or stamp collecting” [15, p. 108] and the mathematics educator William Leonard Schaaf (1898–1992) in his 1978 book [70, page xiii] entitled *Mathematics and Science: An Adventure in Postage Stamps* found that “The postage stamps of the world are, in effect, a mirror of civilization and that multitudes of stamps reflect the impact of mathematics and science on society.” We agree with Schaaf and in this paper we look at stamps depicting (in this order) the founders of Pakistan and (modern) Turkey, a Russian poet, trees from the Holy Land, water birds, tropical fish, antique automobiles, wild mushrooms, classical composers and conductors, government transport vehicles, marine life, flowers, and aircraft.

FIGURE 1.2: Full sheetlet from Pakistan 2005, *Scott* 1063a–1063b.

Depicted on the stamps are Mustafa Kemal Atatürk (1881–1938), founder of the Republic of Turkey as well as its first President, and Quaid-e-Azam Muhammad Ali Jinnah (1876–1948), who is generally regarded as the founder of Pakistan [85]. One stamp depicts only Atatürk. These stamps were issued in 5×4 sheetlets of 20 stamps (10 of each of 2 different stamps) in a chessboard (or checkerboard) pattern (Figure 1.2). There are two distinct 2×2 philatelic Latin squares (PLS) embedded in this sheetlet. The first is, e.g., in the top-left 2×2 corner of Figure 1.2 as shown in Figure 1.1. The other 2×2 PLS, with the two stamps switched is, e.g., in the bottom-right corner of Figure 1.2. Each 2×2 PLS is repeated 6 times (4 times disjointly) in the full sheetlet.

FIGURE 1.3: USSR 1949, *Scott* 1359–1360.

Many countries have issued postage stamps in sheets or sheetlets with a chessboard pattern with various sizes. There are several sheetlets also which depict just 2 stamps each twice and form a 2×2 PLS. An example is the sheetlet (Figure 1.3) issued by USSR in 1949 in celebration of the 150th birth anniversary of Alexander Sergeyevich Pushkin (1799–1837), who is considered to be the greatest Russian poet [85]. The stamp in the top-left corner shows Pushkin in 1822 and the stamp in the top-right corner is based on a portrait by Orest Adamovich Kiprensky (1782–1836) painted in 1827. This is the oldest PLS that we have identified to date. As a referee has pointed out it would be interesting to know who deserves the credit for the (first) use of Latin squares in designing sheets of postage stamps. Canada issued two 10×10 sheetlets each featuring four 5×5 PLS in 1970 (*Scott* 524 & 525) and a 4×4 PLS in 1972 (*Scott* 582–585). The first 3×3 PLS that we have discovered is a pair from the Isle of Man in 1976 (*Scott* 86 & 89).

We note that Pakistan may be the only country to have issued 2×2, 3×3, 4×4 and 5×5 PLS, though we have found only one PLS of each size. In Sections 2 and 3 we study the PLS, respectively, of sizes 3×3 and 4×4. In Section 4 we study a 5×5 PLS from Pakistan and observe that it has the knight's move design as discussed, e.g., by Nissen [56] and by Behrens [10, Tabelle 5a, p. 178 (1956)]. There appear to be no PLS of size $k \times k$ with $k \geq 6$ (from any country), but we have found a 6×5 sheet of stamps (Figure 5.1) from Pakistan depicting 10 kinds of mushrooms (each repeated 3 times); we identify two associated 5×5 Latin squares which do not have the knight's move design and are not orthogonal to each other. These findings lead to what we believe may be the first *philatelic Sudoku puzzle* (Section 6).

2 PHILATELIC LATIN SQUARES OF SIZE 3×3

FIGURE 2.1: Pakistan 1989, *Scott* 704a–704c.

Pakistan issued a 3×3 philatelic Latin square (PLS) in 1989 for the 16th Asian Advertising Congress (Adasia '89) held in Lahore, 18–22 February 1989, see Figure 2.1. The PLS here is a one-step forwards circulant [28,44].

FIGURE 2.2: Israel 1981, *Scott* 798–800.

There are just two possible configurations for a 3×3 PLS: a one-step forwards circulant (Figure 2.1) and a one-step backwards circulant such as the PLS (Figure 2.2) from Israel[3], depicting three kinds of trees from the Holy Land: *arbutus andrachne* (Greek strawberry tree), *cercis siliquastrume* (Judas tree), and *quercus ithaburensis* (Vallonea oak). We may represent these two PLS by, respectively, the 3×3 standard-form circulant matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \tag{2.1}$$

both of which are nonsingular.

[3] Many thanks go to Ian D. Kimmerly for introducing the PLS in Figure 2.2 to us.

3 PHILATELIC LATIN SQUARES OF SIZE 4×4

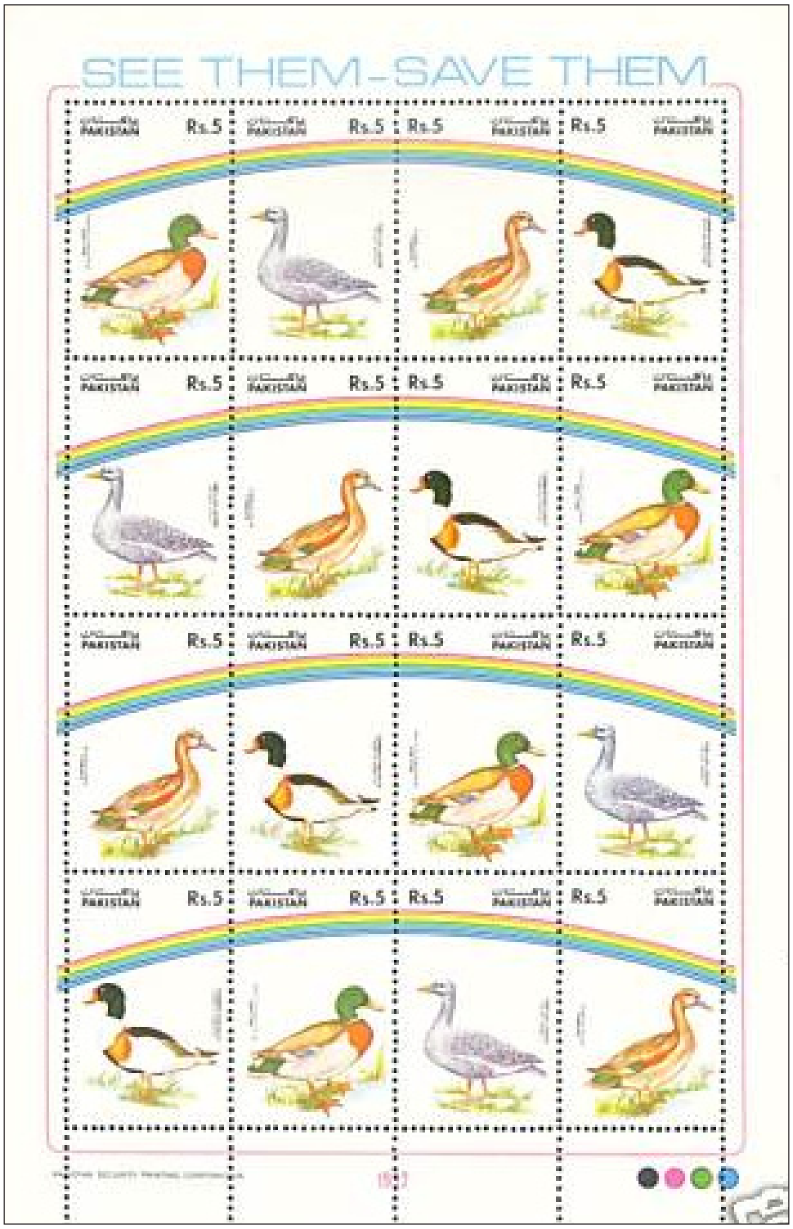


FIGURE 3.1: Pakistan 1992: *Scott* 790a–790d.

We have found a sheetlet of 16 stamps from Pakistan that is almost a 4×4 PLS (Figure 3.1). Issued in 1992 this sheetlet depicts a rainbow and 4 kinds of water birds "See Them – Save Them": (*a*) Gadwall, (*b*) Common Shelduck, (*c*) Mallard, and (*d*) Greylag Goose. Strictly speaking this sheetlet is not a PLS, even though only 4 birds are depicted and each bird 4 times in a Latin-square format—however, the part of the rainbow is different on each of the 4 stamps depicting a particular bird and so each of the 16 stamps is unique. If we ignore the rainbow, this sheetlet is a one-step backwards circulant PLS. This is the only sheetlet of stamps of this type (from any country) that we have found.

If we do not ignore the rainbow then this sheetlet of stamps may be considered to be a philatelic pseudo (or generalized) Graeco-Latin square[4], which we will call a "philatelic rainbow-Latin square". Let us code the 4 parts of the rainbow $1,2,3,4$ and define the two matrices

$$\mathbf{R} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}, \tag{3.1}$$

so that the matrix $\mathbf{R}$ represents the 4 parts of the rainbow depicted in the 4 stamps in each row and the one-step backwards circulant matrix $\mathbf{D}$ represents the ducks. Then the pair

$$(\mathbf{R},\mathbf{D}) = \begin{pmatrix} 1,a & 2,b & 3,c & 4,d \\ 1,b & 2,c & 3,d & 4,a \\ 1,c & 2,d & 3,a & 4,b \\ 1,d & 2,a & 3,b & 4,c \end{pmatrix}, \tag{3.2}$$

is a "rainbow-Latin square". The matrix $\mathbf{R}$ clearly does not define a Latin square and the Latin-square matrix $\mathbf{D}$ does not have an orthogonal Latin-square mate [53], but every one of the 16 pairs (x,y), with $x = 1,2,3,4$ as in $\mathbf{R}$ and $y = a,b,c,d$ as in $\mathbf{D}$, is represented (precisely once) in $(\mathbf{R},\mathbf{D})$.

There are 24 types of Latin squares of size 4×4 in standard-form [53] but only 6 of these 24 types have an orthogonal mate. Of these 6 types we have found PLS only for 4, and for these 4 types of 4×4 Latin squares we have found 60 PLS [53], but we believe that none of these 60, on its own, forms a philatelic Graeco-Latin square[5]. The pair of PLS from

[4] Many thanks go to Jeffrey J. Hunter for suggesting to us that there may exist a philatelic Graeco-Latin square.

[5] There are 2 types of 3×3 Latin squares and these are orthogonal to each other. We have found 18 PLS of the one-step forwards circulant type like the sheetlet from Pakistan (Figure 2.1) and 6 PLS of the one-step backwards circulant type like the sheetlet from Israel (Figure 2.2), but we believe that none of these 24 PLS forms a philatelic Graeco-Latin square: certainly neither the sheetlet from Pakistan (Figure 2.1) nor from Israel (Figure 2.2) is a philatelic Graeco-Latin square or a philatelic rainbow-Latin square.

Pakistan (Figure 2.1) and Israel (Figure 2.2) do, however, form a "philatelic Graeco-Latin square pair" since the 3×3 one-step forwards and backwards circulants are orthogonal to each other.

The Gadwall (*Anas strepera*) is a common and widespread duck of the family *Anatidae* which breeds in the northern areas of Europe and Asia, and central North America. The Common Shelduck (*Tadorna tadorna*) is a common and widespread duck of the genus *Tadorna* which breeds in temperate Eurasia. The Mallard (*Anas platyrhynchos*), probably the best-known and most recognizable of all ducks, is a dabbling duck which breeds throughout the temperate and sub-tropical areas of North America, Europe, Asia, Africa, New Zealand (where it is currently the most common duck species), and Australia. The Mallard is the ancestor of all domestic ducks, except for the few breeds derived from the unrelated Muscovy Duck (*Cairinia moschata*). The Greylag[6] Goose (*Anser anser*) is the type species of the genus *Anser* and is found throughout the Old World and eastwards across Asia to China [85].

FIGURE 3.2: Tristan da Cunha 1981, *Scott* 301a–301d.

The one-step backwards circulant is the most popular kind of 4×4 PLS that we have identified to-date but the one-step forwards circulant (Figure 3.2) is also popular; both 4×4 circulant matrices in standard-form are nonsingular. The stamps[7] in Figure 3.2 are from Tristan da Cunha and depict the Inaccesible Island rail. Tristan da Cunha is a remote

[6] Also spelled Graylag in the United States.

[7] Many thanks go to Götz Trenkler for introducing us to the PLS in Figure 3.2.

volcanic group of islands in the south Atlantic Ocean and is a dependency of the British overseas territory of Saint Helena[8]. The territory consists of the main island Tristan da Cunha, the uninhabited Nightingale Islands and the wildlife reserves of Inaccessible Island and Gough Island. The Inaccessible Island rail (*Atlantisia rogersi*) is found only on Inaccessible Island, and is notable for being the smallest extant flightless bird in the world.

4 LATIN SQUARES OF SIZE 5×5 AND 5×5 SUDOKU

FIGURE 4.1: Pakistan 2004, *Scott* 1045a–1045e.

[8]Napoléon Bonaparte (1769–1821) was exiled to Saint Helena in 1815 and died there on 5 May 1821.

A 5×5 PLS was issued by Pakistan (Figure 4.1) in 2004 for the National Philatelic Exhibition in Lahore on Universal Postal Union Day (October 9). This PLS depicts five kinds of tropical fish: neon tetra, striped gourami, black widow, yellow dwarf cichlid, and tiger barb. The "neon tetra" (*Paracheirodon innesi*) is a freshwater fish of the characin family (family *Characidae*) of order *Characiformes*. The type species of its genus, it is native to blackwater or clearwater streams in southeastern Colombia, eastern Peru, and western Brazil. The giant gourami (*Osphronemus goramy*) is a freshwater fish belonging to the family *Osphronemidae*, also sometimes known as the banded gourami, rainbow gourami, or "striped gourami", native to India. The black tetra (*Gymnocorymbus ternetzi*), also known as the black skirt tetra or "black widow" tetra, is a freshwater fish of the characin family (family *Characidae*), of order *Characiformes* and is native to the Paraguay and Guaporé River basins of southern Brazil, Argentina, and Bolivia. The "yellow dwarf cichlid" (*Apistogramma borellii*) is found in shallow swampy regions along rivers in South America. The "tiger barb" (*Puntius tetrazona*) is a species of tropical freshwater fish belonging to the *Puntius* genus of the minnow family. The natural geographic range reportedly extends throughout the Malay peninsula, Sumatra and Borneo, with unsubstantiated sightings reported in Cambodia [85].

The Latin square in Figure 4.1 has several interesting properties. In particular it is of the type known as a "knight's move" or "Knut Vik" design, as described by Preece [62]:

> Of Latin squares used for crop experiments with a single set of treatments, the earliest examples (published in 1924) are 5×5 squares of the systematic type known as Knut Vik [84] or "knight's move" designs (Knut Vik being a [Norwegian] person, not a Scandinavian translation of "knight's move"!); these are squares where all cells containing any one of the treatments can be visited by a succession of knight's moves (as in chess) and where no two diagonally adjacent cells have the same treatment.

In 1931 Tedin [78] and in 1951 Nissen [56] observe that there are two possible arrangements of the 5×5 Knut Vik design in standard-form: A, B, C, D, E in sequence in the top row. The arrangement KV1 [78, Fig. 1, Arr. 2] has treatment A (in row 2) below C (in row 1) and KV2 [78, Fig. 1, Arr. 1] has A (in row 2) below D (in row 1), see (4.1):

$$\mathrm{KV1} = \begin{pmatrix} A & B & C & D & E \\ D & E & A & B & C \\ B & C & D & E & A \\ E & A & B & C & D \\ C & D & E & A & B \end{pmatrix}, \quad \mathrm{KV2} = \begin{pmatrix} A & B & C & D & E \\ C & D & E & A & B \\ E & A & B & C & D \\ B & C & D & E & A \\ D & E & A & B & C \end{pmatrix}. \tag{4.1}$$

The Latin square KV1 is a two-steps forwards circulant while KV2 is a two-steps backwards circulant. Yates [88], following Tedin [78], notes that KV1 and KV2 are special balanced 5×5 Latin squares in which the treatments are as evenly spaced as possible.

A strict knight's move square is a Latin square such that every two cells which contain the same symbol are joined by a knight's path composed of cells which contain that symbol. Two cells, not in the same row or column, are said to be diagonally adjacent when their distance apart is 2. As Owens [57] has established, there exists a strict knight's move square of order n without diagonal adjacencies if and only if $n = 5$. To within permutations of the symbols and transposition there is only one such square: KV1 or KV2 in (4.1).

The first publication of KV1 or KV2 that we have found is that of KV1 over 300 years ago by Poignard [59, fig. XXIV, p. 74 (1704)]. A few years earlier in 1691, La Loubère[9] [50,51] considered the Latin squares

$$\mathrm{GL1} = \begin{pmatrix} c & e & b & d & a \\ a & c & e & b & d \\ d & a & c & e & b \\ b & d & a & c & e \\ e & b & d & a & c \end{pmatrix}, \quad \mathrm{GL2} = \begin{pmatrix} 0 & \chi & \omega & \phi & \psi \\ \chi & \omega & \phi & \psi & 0 \\ \omega & \phi & \psi & 0 & \chi \\ \phi & \psi & 0 & \chi & \omega \\ \psi & 0 & \chi & \omega & \phi \end{pmatrix}, \tag{4.2}$$

which we call *La Loubère matrices* [77, (3.1)]. The Latin square GL1 is a one-step forwards circulant while GL2 is a one-step backwards circulant. Moreover, GL1 and GL2 are orthogonal to each other in that every ordered pair of symbols from GL1 and GL2 occurs exactly once, and so the pairs (GL1, GL2) and (G2, G1) form Graeco-Latin squares. We believe that the pair (GL2, GL1) considered by La Loubère (1691) is the first Graeco-Latin square to use Greek and Latin letters [77, §3] preceding the seminal work [34, 35] by Leonhard Euler (1707–1783).

The PLS in Figure 4.1 is of type KV2, which is a two-steps backwards circulant; transposing KV2 and recoding the treatments yields KV1 (in standard-form), which is a two-steps forwards circulant as noted by Fisher [41, (book 2) p. 78]:

> In this arrangement the areas bearing each treatment are nicely distributed over the experimental area, so as to exclude all probability that the more important components of heterogeneity should influence the comparison between treatments.

[9] Simon de La Loubère (1642–1729), the French poet and special envoy of King Louis XIV (1638–1715) to Narai the Great (1629–1688), King of the Ayutthaya kingdom of Siam.

The Knut Vik designs KV1 and KV2 are also A-efficient[10] and pandiagonal, in that all diagonals (the two main diagonals and all broken diagonals) contain each of the five treatments precisely once. Moreover no two diagonally-adjacent cells have the same treatment and, as noted by Bailey, Kunert & Martin [6], KV1 and KV2 are the only gerechte designs with this property. As Bailey, Cameron & Connelly [4] point out, in 1956 [10] Walter-Ulrich Behrens[11] (1902–1962) introduced a specialization of Latin squares which he called "gerechte" (Figure 4.2). Behrens [10, Tabelle 5b, p. 178] refers to the Latin square GD2 as a "diagonal layout" (German: *Diagonalanordnung*)—all the elements on the main backwards diagonal are equal (to 1), and [10, Tabelle 5a, p. 178] calls KV1 a "knight's move layout" (German: *Rösselsprunganordnung*) and KV2 [10, Tabelle 8, p. 182] a "gerechte Latin square" (German: *gerechte lateinische Quadrate*).

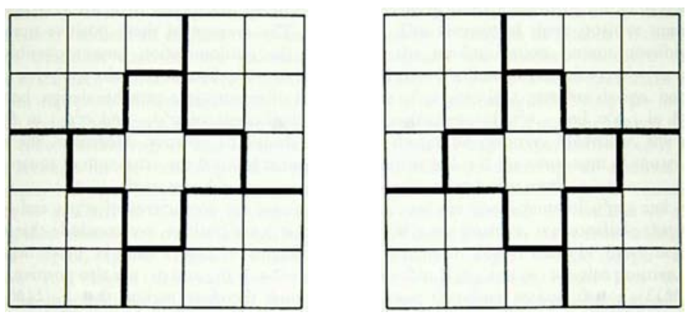


FIGURE 4.2: Gerechte Latin squares GD1 (left panel) and GD2 (right panel).

In a $k \times k$ "gerechte design" the $k \times k$ grid is partitioned into k "regions" $S_1, S_2, \ldots, S_k$, say, each containing k cells of the grid; we are required to place the symbols $1, 2, \ldots, k$ into the cells of the grid in such a way that each symbol occurs once in each row, once in each column, and once in each region. The row and column constraints say that the solution is a Latin square, and the last constraint restricts the possible Latin squares.

Gerechte designs originated in the statistical design of agricultural experiments, where they ensure that treatments are fairly exposed to localized variations in the field containing the experimental plots. The Knut Vik designs KV1 and KV2 are gerechte designs of types GD1 and GD2, respectively. Bailey, Kunert & Martin [5, (1990)] point out that

[10]The A-criterion in the analysis of variance with a Latin square design is equivalent to minimizing: (i) the average variance of treatment-effect estimates, (ii) the average variance of an estimated difference in the treatment-effect estimates, and (iii) the expected value of the treatment sum of squares when there are no treatment differences [55, p. 251].

[11]The well-known Behrens–Fisher problem of comparing means of two populations with possibly unequal variances was first considered by this Walter-Ulrich Behrens [8] and Sir Ronald Aylmer Fisher (1890–1962) [39,40]. For recent results on the Behrens–Fisher problem see Ruben [69].

> Gerechte designs are row and column designs which have an additional blocking structure formed by spatially compact regions. They were popularized in Germany in 1956 by Behrens [9, 10]. We use the German word "gerechte" for these designs, as there is no simple word in English to express the idea of "fair-allocation", and because the designs originated in Germany, and do not appear to have spread from there. Behrens's idea was to have a class of designs that were efficient, as good systematic designs are, but that allow some randomization.

Solutions to 9×9 Sudoku puzzles are examples of gerechte Latin squares, where $k = 9$ and the regions $S_1, S_2, \ldots, S_9$ are the nine contiguous 3×3 regions (submatrices or boxes). Colbourn, Dinitz & Wanless [22] define a completed 9×9 Sudoku puzzle as a $(3,3)$-*Sudoku Latin square*. While Sudoku puzzles have become popular only very recently, Sudoku's French ancestors from the late 19th century are discussed by Boyer [18, 19]; see also Bailey, Cameron & Connelly [4] for an excellent survey of the connections between Latin squares and Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes. There are many books on Sudoku, at many levels. We particularly like the book *How to Solve Sudoku,* by Robin Wilson [87]. Most books on Sudoku concentrate on the 9×9 puzzle, but *Absolute Sudoku* [91] has 4×4 and 6×6, as well as 9×9 puzzles. *Sudoku Variants* [23] has many different shapes of Sudoku puzzles including 15 puzzles of size 6×6 with various different "irregular" blocking patterns. We found very few books that include the smaller-size Sudoku puzzles.

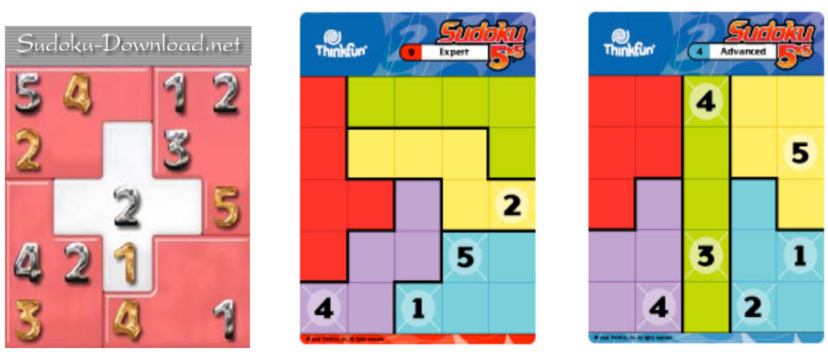


FIGURE 4.3: 5×5 Sudoku puzzle proposed by Maack [54] (left panel) and ThinkFun [79] (centre and right panels).

Very little seems to have been written about 5×5 Sudoku puzzles: the only websites that we have found with such puzzles are by Maack [54] and ThinkFun [79]. The Sudoku-Download website created by A. Maack [54] contains Sudoku puzzles and solutions of many sizes, including size 5×5 using the gerechte design GD1 (Figure 4.2, left panel), which Maack calls “cross”, see Figures 4.2 (left panel) and 4.3 (left panel). Maack [54] gives 240 Sudoku puzzles and solutions of size 5×5, but none of the solutions are in standard-form: $1, 2, 3, 4, 5$ in sequence in the top row. The gerechte designs used by ThinkFun for 5×5 Sudoku puzzles differ from those used by Maack, see Figure 4.3 (centre and right panels): in these examples there is no central “cross” region and the regions all have different shapes. In the second ThinkFun example (Figure 4.3, right panel), one region is an entire column: the centre column.

The PLS from Pakistan shown in Figure 4.1 is the only PLS (from any country) that we have found of type KV2, and we have found just one PLS (from any country) of type KV1 (Figure 4.4). Depicted are these antique automobiles from the late 19th and (very) early 20th century (from left to right in the first row): (1) 1893 Duryea, (2) 1894 Haynes, (3) 1898 Columbia, (4) 1899 Winton, (5) 1901 White.

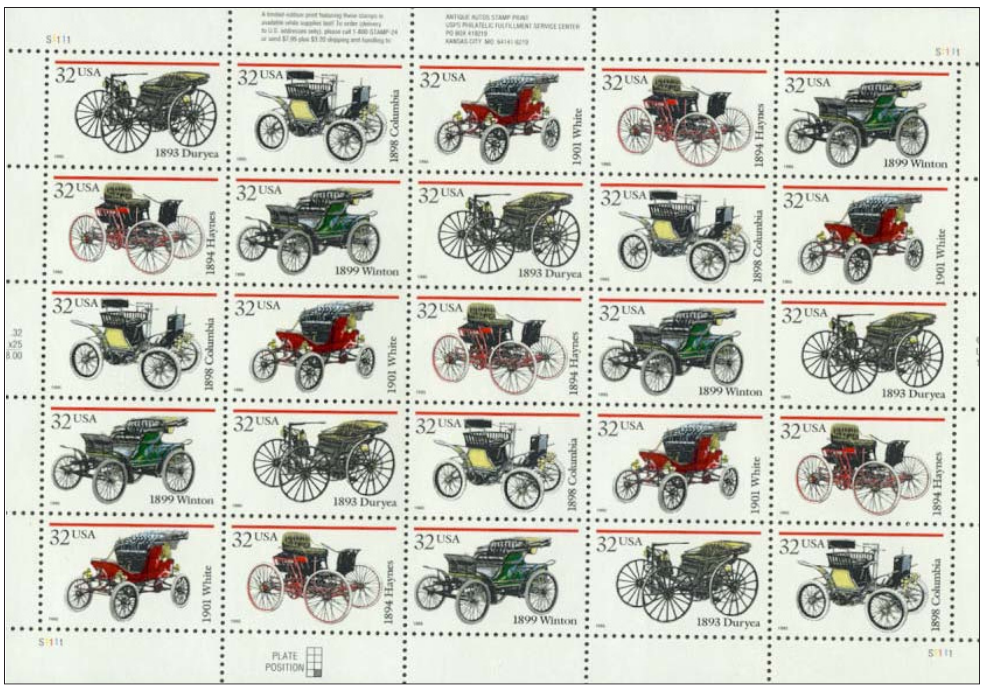


FIGURE 4.4: USA 1995, *Scott* 3023v.

5 TWO MORE 5×5 LATIN SQUARES ASSOCIATED WITH POSTAGE STAMPS FROM PAKISTAN

FIGURE 5.1: Pakistan 2005, *Scott* 1071.

We have found no other PLS from Pakistan of any size but we have found a sheet of 6×5 stamps from Pakistan 2005, *Scott* 1071, depicting 10 different mushrooms (Figure 5.1) each with 1 stamp repeated 3 times. If we number the mushrooms $1, 2, \ldots, 10$ (Table 5.1), then the PLS in Figure 5.1 may be represented as in Figure 5.2. Table 5.1 gives the Latin names for the various mushrooms, together with some comments concerning where they are found[12] and their edibility; some French and German names are also included. There seems to be no obvious reason why the mushrooms have been grouped the way they have. Eight of the types of mushroom are edible, some even considered choice [68].

TABLE 5.1: Mushrooms depicted in stamps from Pakistan 2005, *Scott* 1071 (Figure 5.1).

1. *Lepiota procera* (parasol mushroom, coulemelle, lépiote élevée, Riesenschirmling, Riesenschirmpilz, Parasolpilz); found in North America, Europe; edibility: choice [68], eagerly sought after for food [46, p. 83],

2. *Tricholoma gambosum* (*Calocybe gambosum* [46, p. 88], St. George's mushroom, mousseron, Maipilz), found in North America, Europe; edibility: choice [68], a delicacy, especially when fried in butter [85],

3. *Amanita caesarea* (Caesar's mushroom, amanite des césars, oronge, Kaiserling); found in North America, Europe; edibility: choice [68],

4. *Cantharellus cibarius* (golden chanterelle, girolle, Eierschwamm, Pfifferling); found in North America, Europe; edibility: choice [68], excellent with scrambled eggs [76],

5. *Boletus luridus* (lurid bolete, cep, bolet comestible, netzstieliger Hexenröhrling); found in North America; edibility: poisonous/suspect [68],

6. *Morchella vulgaris* (morel, morille blonde, morille commune, morille grise, gemeine Morchel); found in North America, Europe; edibility: choice [68],

7. *Amanita vaginata* (grisette mushroom, coucoumelle grise, grauer Scheidenstreifling); found in North America, Europe; edibility: poisonous/suspect [68],

8. *Agaricus arvensis* (horse mushroom, psalliote des jachères, agaric des jachères, boule de neige, Schafchampignon); found in North America, Europe; edible [68], "much prized by farmers and gypsies for generations, one of the most delicious edible fungi" [85],

9. *Coprinus comatus* (lawyer's wig, shaggy ink cap, shaggy-mane mushroom, coprin chevelu, escumelle, Schopftintling); found in North America, Europe; edible [68],

10. *Clitocybe geotropa* (trooping funnel, lépiste nu, pied bleu, Mönchskopf); found in North America, Europe; edible [68].

[12] We do not know if any of these mushrooms are to be found in Pakistan and/or are used in Pakistani cooking.

1	2	3	4	5
6	7	8	9	10
5	4	1	2	3
7	9	10	6	8
4	5	2	3	1
8	10	6	7	9

FIGURE 5.2: Numerical representation for the 6×5 sheet of stamps shown in Figure 5.1, using the coding given in Table 5.1.

1	2	3	4	5
2	3	5	1	4
3	1	4	5	2
4	5	2	3	1
5	4	1	2	3

6	7	8	9	10
7	9	10	6	8
8	10	6	7	9
9	8	7	10	6
10	6	9	8	7

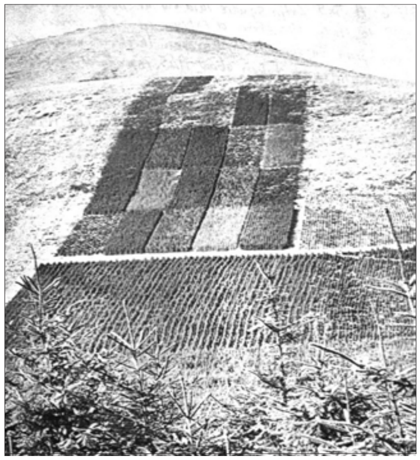
FIGURE 5.3: Two 5×5 Latin squares in standard-form generated from Figure 5.2.

Probably the most famous Latin square in the design of experiments is the 5×5 Latin square[13] in the Beddgelert[14] Forest in north Wales (Figure 5.4, left panel). This experiment was designed by R. A. Fisher in 1926 and laid out in 1929 [4, p. 387] to study the effect of exposure on 5 species of trees, which were planted according to the Latin square[15] shown in Figure 5.4 (right panel): (1) Japanese larch, (2) Sitka spruce, (3) Norway spruce/European larch: 50/50, (4) Sitka spruce/*Pinus contorta:* 50/50, (5) Sitka spruce/Japanese larch: 50/50.

[13] Many thanks go to Gavin J. S. Ross for drawing our attention to this Latin square.

[14] Beddgelert is a village in Gwynedd, Wales, lying in Snowdonia. In Welsh Beddgelert is written "Beđgelert", pronounced locally "Bethgelart". Older spellings include "Bettgelert", "Beth Kellarth" and "Beth Kelert" [65, p. 567]. Beđgelert is Welsh for "Gelert's grave" and it is rumoured to be named after a dog named Gelert. The so-called grave is now a tourist attraction, but there is [apparently?] no truth in the Gelert legend [71, p. 13], which was fostered by local innkeeper David Prichard during the 19th century.

[15] See Box [16, Plates 6 & 7 (between pp. 256 & 257)], Preece [62, pp. 931–932], Senn [71, p. 53] and Bailey, Cameron & Connelly [4, pp. 386–387]; see also [77].



1	2	3	4	5
5	3	1	2	4
2	5	4	3	1
4	1	2	5	3
3	4	5	1	2

FIGURE 5.4: (left panel) Five species of trees in Latin-square format in the Beddgelert Forest in north Wales laid out in 1929 and photographed in 1952 [4, p. 387], and (right panel) Latin square in "gerechte Fisher-trees format".

We notice that the Latin square layout of the trees (Figure 5.4, right panel) is not in gerechte format GD1 or GD2 (Figure 4.2), but is similar to the ThinkFun layout shown in Figure 4.3 (right panel), except that here two regions are complete rows while in Figure 4.3 (right panel) just one region is a complete column. We will refer to the Latin square in Figure 5.4 (right panel) as being in "gerechte Fisher-trees format". If we rearrange rows 2–5 in Figure 5.3 (left panel) then we find just two rearrangements which are Latin squares in gerechte Fisher-trees format, see Figure 5.5. No other rearrangement[16] of rows 2–5 and no rearrangement of rows 2–5 of Figure 5.4 yields this format.

1	2	3	4	5
3	1	4	5	2
2	3	5	1	4
4	5	2	3	1
5	4	1	2	3

1	2	3	4	5
4	5	2	3	1
2	3	5	1	4
3	1	4	5	2
5	4	1	2	3

FIGURE 5.5: Two 5×5 Latin squares in gerechte Fisher-trees format generated from Figure 5.3.

[16]One rearrangement (only) of rows 2–5 in Figure 5.3 (left panel) is in a format which is the transpose of the gerechte Fisher-trees format.

6 PHILATELIC SUDOKU PUZZLES

We may use the 6×5 sheetlet of mushroom-stamps from Pakistan (Figure 5.1) to create a 10×10 Sudoku puzzle—our first "philatelic Sudoku puzzle"! We present six such puzzles (we have found only six[17], from any country): the 10×10 puzzle S_1 based on the mushroom-stamps from Pakistan (Figure 5.1), an 8×8 puzzle S_2 based on stamps featuring classical composers and conductors from the USA (Figure 6.2.1 [7]), a 6×6 puzzle S_3 based on transport-stamps from Hong Kong (Figure 6.3.1), a 12×12 puzzle S_4 based on stamps featuring marine life from Abkhazia (Figure 6.4.3), a 10×10 puzzle S_5 with stamps from the USA depicting flowers (Figure 6.5.1), and a 10×10 puzzle S_6 with stamps from the USA depicting aircraft. The solutions we find for puzzles $S_1 - S_5$ (but not S_6) are, in reduced-form, all block-Latin.

Our philatelic Sudoku puzzle (PSP) is based on a sheetlet of size $r \times c$ $(r \neq c)$ featuring $s \geq \max(r,c)$ distinct stamps arranged in a "Latin rectangle" (we do not require that rc/s be an integer); the puzzle here is to find an $s \times s$ Latin square in which the Latin rectangle defining the sheetlet is a subregion and some blocking within the subregion is involved as with the popular "regular" 9×9 Sudoku puzzle and our six philatelic Sudoku puzzles. We will let b denote the block size and so $b = 9$ in regular Sudoku. We will say that the puzzle is "composite" when the solution can be blocked in two different ways as with puzzles S_3 and S_4.

TABLE 6.1: Philatelic Sudoku puzzles.

PSP	country	year	topic	*Scott*	r	c	s	b	g%	ρ	ν
S_1	Pakistan	2005	mushrooms	1071	6	5	10	5	30	9	1
S_2	USA	1997	musicians	3165v	5	4	8	4	$37\frac{1}{2}$	6	2
S_3	Hong Kong	2006	transport vehicles	1219v	6	3	6	3	50	6	0
S_4	Abkhazia	2006	marine life	*BiStamp* 252	8	3	12	4	$16\frac{2}{3}$	6	6
S_5	USA	2007	flowers	4185v	2	10	10	2	20	7	3
S_6	USA	2005	aircraft	3925v	5	4	10	10	20	10	0

Table 6.1 summarizes the basic information of our six philatelic Sudoku puzzles. We have included the percentage $g = 100rc/s^2$ of "givens", i.e., numbers given; in regular 9×9 Sudoku about 30 cells are often given[18] out of 81, and so there $g \sim 37\%$; for our philatelic Sudoku puzzles g ranges from just under 17% all the way to 50%. We have also included

[17] We look forward to finding other philatelic Sudoku puzzles.

[18] In a sample of 180 (60 "hard", 60 "medium" and 60 "easy") regular 9×9 Sudoku puzzles from Maack [54], we found an average of 29.09 cells filled (35.92%): for the 60 "hard" an average of 23.87 (29.47%), for the 60 "medium" 29.15 cells filled (35.99%), and for the 60 "easy" 34.27 (42.30%).

the rank ρ and the nullity ν of the $s \times s$ Latin-square matrix solution to the puzzle, with $s = \rho + \nu$. We find it interesting to note that $6 \leq s \leq 12$, $6 \leq \rho \leq 10$, and $0 \leq \nu \leq 6$.

When no blocking is involved then we call the philatelic Sudoku puzzle a "philatelic Youden[19] puzzle". While we have found only five philatelic Sudoku puzzles, we have found many philatelic Youden puzzles, most of which have $|r - c| = 1$ and $s = \max(r, c)$[20]. Deleting one row (or column) from a philatelic Latin square leaves a "philatelic Latin rectangle" in which every pair of stamps appears together in the same column (or row) an equal number of times. Youden [89, 90] pointed out that more than one row can sometimes be deleted to leave a Latin rectangle with this property. Such Latin rectangles [61] have come to be popularly known as "Youden squares", see, e.g., Preece [63], but are called "Youden rectangles" by Preece [60]. We plan to study philatelic Youden puzzles in a further paper.

6.1 Philatelic Sudoku puzzle S_1: mushroom-stamps from Pakistan

Our first philatelic Sudoku puzzle S_1 is based on the 6×5 sheetlet of stamps from Pakistan featuring 10 different varieties of mushrooms (Figure 5.1), and is "blocked" in the Maack "brickwall" pattern (Figure 6.1) except that here, in addition, the numbers $1, 2, 3, 4, 5$ and the numbers $6, 7, 8, 9, 10$ are restricted to occur in alternate rows, see Figure 6.1.3. Maack [54] gives 10×10 Sudoku puzzles of 5 distinct gerechte patterns, see Figure 6.1.1, and also uses the brickwall pattern in 8×8 Sudoku puzzles, see Figure 6.1.2.

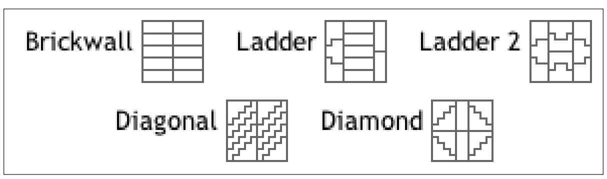


FIGURE 6.1.1: Five different Maack 10×10 Sudoku puzzle formats.

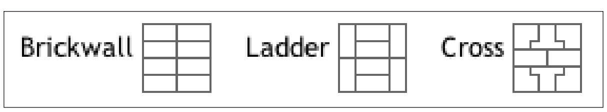


FIGURE 6.1.2: Three different Maack 8×8 Sudoku puzzle formats.

[19] The American chemical engineer and statistician William John Youden (1900–1971) [33] is considered in the *Handbook of Combinatorial Designs* [21, p. 21] to have been one of the four main contributors to design theory from antiquity to 1950 (born between 1890 and 1902).

[20] We have found 12 philatelic Youden puzzles from the USA with $|r - c| = 1$ and $s = 5$ including one featuring Gulf Coast lighthouses, issued on 23 July 2009.

The puzzle S_1 here (Figure 6.1.3, left panel) is to create a 10×10 Latin square with the extra conditions that the numbers $1, 2, 3, 4, 5$ appear, in some order, in each of the regions coloured pale blue (Figure 6.1.3), and that the numbers $6, 7, 8, 9, 10$ appear, in some order, in each of the other regions. Our solution (using Figures 5.2–5.4) to puzzle S_1 is shown in Figure 6.1.3 (right panel, cells with "givens" are in brown) and uses the two 5×5 Latin square matrices:

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} 6 & 7 & 8 & 9 & 10 \\ 7 & 9 & 10 & 6 & 8 \\ 8 & 10 & 6 & 7 & 9 \\ 9 & 8 & 7 & 10 & 6 \\ 10 & 6 & 9 & 8 & 7 \end{pmatrix}, \tag{6.1.1}$$

see also Figure 5.3.

1	2	3	4	5					
6	7	8	9	10					
5	4	1	2	3					
7	9	10	6	8					
4	5	2	3	1					
8	10	6	7	9					

1	2	3	4	5	6	7	8	9	10
6	7	8	9	10	1	2	3	4	5
5	4	1	2	3	10	6	9	8	7
7	9	10	6	8	2	3	5	1	4
4	5	2	3	1	9	8	7	10	6
8	10	6	7	9	3	1	4	5	2
2	3	5	1	4	7	9	10	6	8
9	8	7	10	6	4	5	2	3	1
3	1	4	5	2	8	10	6	7	9
10	6	9	8	7	5	4	1	2	3

FIGURE 6.1.3: Philatelic Sudoku puzzle S_1 based on mushroom-stamps from Pakistan (Figure 5.1), with solution (right panel).

If the rows in Figure 6.1.3 (right panel) are rearranged so that the numbers in column 1 are in ascending order, and so the 10×10 Latin square is now in reduced-form, then it may be represented by the 10×10 Latin square matrix $\mathbf{B}_1$ in 2×2 block-Latin format:

$$\mathbf{B}_1 = \begin{pmatrix} \mathbf{L}_1 & \mathbf{M}_1 \\ \mathbf{M}_1 & \mathbf{L}_1 \end{pmatrix}, \tag{6.1.2}$$

where the 5×5 Latin square matrices $\mathbf{L}_1$ and $\mathbf{M}_1$ are as defined in (6.1.1). The rank and a basis for the null space of $\mathbf{B}_1$ are easily found[21]. Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{L} & \mathbf{M} \\ \mathbf{M} & \mathbf{L} \end{pmatrix}, \tag{6.1.3}$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ in (6.1.3) are both $p \times q$, neither necessarily square. Then we have the "2×2 block-Latin rank formula"

$$\operatorname{rank}(\mathbf{B}) = \operatorname{rank}(\mathbf{L}+\mathbf{M}) + \operatorname{rank}(\mathbf{L}-\mathbf{M}), \tag{6.1.4}$$

given by Tian & Styan [80, p. 115]; see also Zhang [92, Problem 2.40, pp. 35, 140–141]. Moreover, let the columns of the $q \times s$ matrix $\mathbf{W}$ span the (column) null space of $\mathbf{L}+\mathbf{M}$ and let the columns of the $q \times t$ matrix $\mathbf{X}$ span the (column) null space of $\mathbf{L}-\mathbf{M}$. Then

$$\mathbf{B}\begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{W} & -\mathbf{X} \end{pmatrix} = \begin{pmatrix} \mathbf{L} & \mathbf{M} \\ \mathbf{M} & \mathbf{L} \end{pmatrix}\begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{W} & -\mathbf{X} \end{pmatrix} = \begin{pmatrix} (\mathbf{L}+\mathbf{M})\mathbf{W} & (\mathbf{L}-\mathbf{M})\mathbf{X} \\ (\mathbf{L}+\mathbf{M})\mathbf{W} & -(\mathbf{L}-\mathbf{M})\mathbf{X} \end{pmatrix} = \mathbf{0}, \tag{6.1.5}$$

and so the columns of the $2q \times (s+t)$ matrix

$$\mathbf{C} = \begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{W} & -\mathbf{X} \end{pmatrix} \tag{6.1.6}$$

span[22] the (column) null space of $\mathbf{B}$. Using (6.1.4) we find that

$$\operatorname{rank}(\mathbf{B}_1) = \operatorname{rank}(\mathbf{L}_1+\mathbf{M}_1) + \operatorname{rank}(\mathbf{L}_1-\mathbf{M}_1) = 5+4 = 9, \tag{6.1.7}$$

and so the 10×10 matrix representing the solution (Figure 6.1.3) to the 10×10 philatelic Sudoku puzzle S_1 is singular with rank equal to 9; the (column) null space has dimension 1 and is spanned by $\left(\begin{smallmatrix} \mathbf{x}_1 \\ -\mathbf{x}_1 \end{smallmatrix}\right)$, where $\mathbf{x}_1 = (-4,1,1,1,1)'$ spans the null space of $\mathbf{L}_1 - \mathbf{M}_1$.

[21] As a referee has pointed out it is easy to use a computer-algebra program to find the rank and a basis for the null space of $\mathbf{B}_1$ but our method here allows this arithmetic to be completed even without a computer.

[22] It is easy to see that the matrix $\mathbf{C}$ in (6.1.6) has full column rank.

6.2 Philatelic Sudoku puzzle S_2: stamps from the USA featuring classical composers and conductors

FIGURE 6.2.1: USA 1997, *Scott* 3165v.

While the 10×10 philatelic Sudoku puzzle S_1 based on the 6×5 sheetlet of mushroom-stamps from Pakistan is the only 10×10 philatelic Sudoku puzzle that we have found (from any country), we have found a sheetlet of 5×4 stamps (Figure 6.2.1) from the USA featuring eight classical composers and conductors in the "Legends of American music series" and which yields the 8×8 philatelic Sudoku puzzle S_2 (Figure 6.2.1) in Maack–brickwall format (Figure 6.1.2). The 8 classical composers and conductors depicted in Figure 6.2.1 are (1) Leopold Stokowski (1882–1977), (2) Arthur Fiedler (1894–1979), (3) George Szell (1897–1970); (4) Eugene Ormandy (1899–1985), (5) Samuel Barber (1910–1981), (6) Ferde Grofé (1892–1972), (7) Charles Ives (1874–1954), (8) Louis Moreau Gottschalk (1829–1869).

1	2	3	4				
5	6	7	8				
3	4	1	2				
7	8	5	6				
4	3	2	1				

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
3	4	1	2	7	8	5	6
7	8	5	6	3	4	1	2
4	3	2	1	6	7	8	5
6	7	8	5	4	3	2	1
2	1	4	3	8	5	6	7
8	5	6	7	2	1	4	3

FIGURE 6.2.2: Philatelic Sudoku puzzle S_2 based on Figure 6.2.1, with solution (right panel).

The puzzle (Figure 6.2.2, left panel) is to create an 8×8 Latin square with the extra conditions that the numbers $1,2,3,4$ appear, in some order, in each of the regions coloured pale blue, and that the numbers $5,6,7,8$ appear, in some order, in each of the other regions. Our solution to puzzle S_2, using the two 4×4 Latin squares in (6.2.2) below is shown in Figure 6.2.2 (right panel), the cells with "givens" are in brown.

If the rows in Figure 6.2.2 (right panel) are rearranged so that the numbers in column 1 are in ascending order, and so the 8×8 Latin square is now in reduced-form, then it may be represented by the 8×8 Latin square matrix $\mathbf{B}_{S_2}$ in block-Latin format, see also (6.1.2) above:

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{L}_2 & \mathbf{M}_2 \\ \mathbf{M}_2 & \mathbf{L}_2 \end{pmatrix}, \tag{6.2.1}$$

where the 4×4 matrices $\mathbf{L}_2$ and $\mathbf{M}_2$ are defined as:

$$\mathbf{L}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 5 \\ 7 & 8 & 5 & 6 \\ 8 & 5 & 6 & 7 \end{pmatrix}. \tag{6.2.2}$$

Moreover $\mathbf{L}_2$ is "criss-cross" in that all elements of the main-forwards diagonal are equal as are all elements of the main-backwards diagonal, and is "centro-symmetric" [12, p. 181] in that reversing the columns and the rows leaves $\mathbf{L}_2$ unchanged. The Latin square matrix $\mathbf{M}_2$ is the one-step backwards circulant. We also find it interesting that both matrices $\mathbf{L}_2$ and $\mathbf{M}_2$ in (6.2.2) are themselves 2×2 block-Latin (with 2×2 block submatrices). Using the 2×2 block-Latin rank formula (6.1.4) above repeatedly, it is easy to see that the matrix representing the solution (Figure 6.2.2, right panel) to the 8×8 philatelic Sudoku puzzle S_2 is singular with rank equal to 6; moreover the (column) null space is spanned by the (columns of the) matrix $\begin{pmatrix} \mathbf{X}_2 \\ -\mathbf{X}_2 \end{pmatrix}$, where

$$\mathbf{X}_2 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2.3}$$

6.3 Philatelic Sudoku puzzle S_3: transport-stamps from Hong Kong

Our third philatelic Sudoku puzzle S_3 is based on a 6×3 sheetlet (Figure 6.3.1, below) of stamps from Hong Kong featuring 6 different modes of government transport, which we code (Figure Kronecker-sum) as follows: (1) Correctional Services Department: Security Bus, (2) Fire Services Department: Hydraulic Platform, (3) Hong Kong Police Force: Versatile Traffic Patrol Motorcycle, (4) Customs and Excise Department: Mobile X-ray Vehicle Scanning System, (5) Government Flying Service: Super Puma Helicopter, (6) Immigration Department: Launch. Then we may represent the sheetlet as shown in Figure 6.3.2 (left panel).

FIGURE 6.3.1: Hong Kong 2006, *Scott* 1219v.

1	2	3
4	5	6
2	3	1
5	6	4
3	1	2
6	4	5

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

1	2	3	4	5	6
4	5	6	1	2	3
2	3	1	5	6	4
5	6	4	2	3	1
3	1	2	6	4	5
6	4	5	3	1	2

FIGURE 6.3.2: Philatelic Sudoku puzzle S_3 (left panel) with solution (right panel).

To solve this puzzle S_3 we first sort so that the elements in column 1 are in ascending order (Figure 6.3.2, center panel, shaded part). We then form a 6×6 Latin square which is 2×2 block-Latin with 3×3 blocks (Figure 6.3.2, center panel). We then rearrange the rows back into the original sequence to find the solution (Figure 6.3.2, right panel). It is easy to see that the matrix representing the solution has rank equal to 4 and (column) null space spanned by the columns of $\begin{pmatrix} \mathbf{X}_3 \\ -\mathbf{X}_3 \end{pmatrix}$, where $\mathbf{X}_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$.

The reduced-form solution to puzzle S_3 (Figure 6.3.2, center panel) may be represented by the matrix

$$\mathbf{B}_3 = \begin{pmatrix} \mathbf{L}_3 & \mathbf{M}_3 \\ \mathbf{M}_3 & \mathbf{L}_3 \end{pmatrix}, \tag{6.3.1}$$

where

$$\mathbf{L}_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mathbf{M}_3 = \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{pmatrix} = \mathbf{L}_3 + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}. \tag{6.3.2}$$

The 2×2 block-Latin representation (6.3.1) and the fact that all the elements of $\mathbf{M}_3 - \mathbf{L}_3$ are equal (to 3) allow us to rewrite the matrix $\mathbf{B}_3$ in (6.3.1) as a "generalized Kronecker-sum"

$$\mathbf{B}_3 = \mathbf{E}_{2\times 2} \otimes \mathbf{K}_{3\times 3} + 3\mathbf{K}_{2\times 2} \otimes \mathbf{E}_{3\times 3} - 3\mathbf{E}_{2\times 2} \otimes \mathbf{E}_{3\times 3} = \mathbf{B}_3^{(2,3)}, \tag{6.3.3}$$

say, where $\otimes$ denotes Kronecker product, $\mathbf{E}_{p\times q}$ is the $p \times q$ matrix with every element equal to 1 and

$$\mathbf{K}_{2\times 2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{K}_{3\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \mathbf{L}_3. \tag{6.3.4}$$

The “Kronecker sum” $\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{B}$, where $\mathbf{A}$ is $m \times m$ and $\mathbf{B}$ is $n \times n$, see, e.g., Bernstein [12, p. 443].

The formula (6.3.3) suggests that the matrix

$$\mathbf{B}_3^{(3,2)} = \mathbf{E}_{3\times3} \otimes \mathbf{K}_{2\times2} + 2\mathbf{K}_{3\times3} \otimes \mathbf{E}_{2\times2} - 2\mathbf{E}_{3\times3} \otimes \mathbf{E}_{2\times2}, \tag{6.3.5}$$

also provides a reduced-form solution to philatelic Sudoku puzzle S_3 and this is indeed so, but with the “column-wise” coding (Figure 6.3.3): (1) Correctional Services Department: Security Bus, (2) Customs and Excise Department: Mobile X-ray Vehicle Scanning System, (3) Fire Services Department: Hydraulic Platform, (4) Government Flying Service: Super Puma Helicopter, (5) Hong Kong Police Force: Versatile Traffic Patrol Motorcycle, (6) Immigration Department: Launch.

1	3	5
2	4	6
3	5	1
4	6	2
5	1	3
6	2	4

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	3	4
6	5	2	1	4	3

FIGURE 6.3.3: Philatelic Sudoku puzzle S_3 with column-wise coding and reduced-form solution (right panel).

A generalized Kronecker-sum of the type given by (6.3.3) and (6.3.5) was used by Rogers, Loly & Styan [67] in a study of composite magic squares. A formulation of this generalized Kronecker-sum type is not possible for philatelic Sudoku puzzles S_1 and S_2 since there the $\mathbf{L}$ and $\mathbf{M}$ matrices do not differ by a scalar multiple of $\mathbf{E}$ as in (6.3.2). When $\mathbf{L} - \mathbf{M}$ is a scalar multiple of $\mathbf{E}$ we will say that the associated $\mathbf{B}$-matrix defines a “composite block-Latin square”[23] and we have a “composite philatelic Sudoku puzzle”. As we will show in the next section, our fourth philatelic Sudoku puzzle S_4 is a composite block-Latin square, again with two representations of the generalized Kronecker-sum type.

[23] The term “composite Latin square” has been used to mean “Sino-Graeco-Latin square” [1].

6.4 Philatelic Sudoku puzzle S_4: stamps from Abkhazia featuring marine life

Our fourth philatelic Sudoku puzzle S_4 is based on an 8×3 sheetlet (Figure 6.4.1) of stamps from Abkhazia[24] featuring 12 different kinds of "sea animals" or marine life[25], which we code row-wise (top 4×3 block):

(1) *Henricia sanguinolenta* (blood sea star),
(2) *Oreaster nodosus* (*Protoreaster nodosus,* horned or chocolate-chip sea star),
(3) *Ceramaster patagonicus* (orange-cookie star),
(4) *Eulimnogammarus virgatus* (variety of Lake Baikal amphipod [81, 82, 86]),
(5) *Pallasea bicornis* (*Propachygammarus bicornis*, variety of Lake Baikal amphipod [81, 82, 86]),
(6) *Cyphocaris richardi* (Antarctic[26] deep-sea amphipod [27, 82, 72]),
(7) *Stylocidaris affinis* (cidaroid sea-urchin),
(8) *Phyllacanthus imperialis* (lance urchin),
(9) *Heterocentrotus mammillatus* (slate-pencil urchin),
(10) *Sclerocrangon salebrosa* (Bering shrimp),
(11) *Stenopus hispidus*[27] (banded-cleaning shrimp, barber-pole shrimp,
(12) *Lebbeus sp.*[28]

We find that the philatelic Sudoku puzzle S_4 admits the two reduced-form generalized Kronecker-sum solutions (Figure 6.4.3)

$$\mathbf{B}_4^{(4,3)} = \mathbf{E}_{4\times 4} \otimes \mathbf{K}_{3\times 3} + 3\mathbf{K}_{4\times 4} \otimes \mathbf{E}_{3\times 3} - 3\mathbf{E}_{4\times 4} \otimes \mathbf{E}_{3\times 3}, \tag{6.4.1}$$

$$\mathbf{B}_4^{(3,4)} = \mathbf{E}_{3\times 3} \otimes \mathbf{K}_{4\times 4} + 4\mathbf{K}_{3\times 3} \otimes \mathbf{E}_{4\times 4} - 4\mathbf{E}_{3\times 3} \otimes \mathbf{E}_{4\times 4}, \tag{6.4.2}$$

where the 3×3 and 4×4 one-step backwards circulants

$$\mathbf{K}_{3\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mathbf{K}_{4\times 4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}. \tag{6.4.3}$$

[24] Abkhazia is a disputed region on the eastern coast of the Black Sea. Since its declaration of independence from Georgia in 1991 during the Georgian–Abkhaz conflict, it is governed by the partially-recognized Republic of Abkhazia [85].

[25] Many thanks go to Torsten Bernhardt, Mikhail Daneliya, and Risto Väinölä for their help in identifying these 12 different kinds of marine life.

[26] Chevreux [20] reports finding two specimens in the water near the Azores, 36°17' N, 28°53' W.

[27] *Stenopus hispidus* is a shrimp-like decapod crustacean, belonging to the infraorder *Stenopodidea.* Although it looks like a shrimp and has the word shrimp in its name, it is not a true shrimp [85].

[28] The species is not given on the stamp but we conjecture that this crustacean may be the *Lebbeus groenlandicus* (spiny lebbeid) [24].

FIGURE 6.4.1: Abkhazia 2006, *BiStamp*[29] 252.

[29] Stamps from Abkhazia are catalogued by *BiStamp* [14] but not by Ŝcott [47].

1	2	3
4	5	6
7	8	9
10	11	12
3	1	2
6	4	5
9	7	8
12	10	11

1	5	9
2	6	10
3	7	11
4	8	12
9	1	5
10	2	6
11	3	7
12	4	8

FIGURE 6.4.2: Row-wise (left panel) and column-wise (right panel) coding and blocking of philatelic Sudoku puzzle S_4 based on Abkhazia "sea animals" stamps (Figure 6.4.2).

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	8	9	7	11	12	10
3	1	2	6	4	5	9	7	8	12	10	11
4	5	6	7	8	9	10	11	12	1	2	3
5	6	4	8	9	7	11	12	10	2	3	1
6	4	5	9	7	8	12	10	11	3	1	2
7	8	9	10	11	12	1	2	3	4	5	6
8	9	7	11	12	10	2	3	1	5	6	4
9	7	8	12	10	11	3	1	2	6	4	5
10	11	12	1	2	3	4	5	6	7	8	9
11	12	10	2	3	1	5	6	4	8	9	7
12	10	11	3	1	2	6	4	5	9	7	8

1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	1	6	7	8	5	10	11	12	9
3	4	1	2	7	8	5	6	11	12	9	10
4	1	2	3	8	5	6	7	12	9	10	11
5	6	7	8	9	10	11	12	1	2	3	4
6	7	8	5	10	11	12	9	2	3	4	1
7	8	5	6	11	12	9	10	3	4	1	2
8	5	6	7	12	9	10	11	4	1	2	3
9	10	11	12	1	2	3	4	5	6	7	8
10	11	12	9	2	3	4	1	6	7	8	5
11	12	9	10	3	4	1	2	7	8	5	6
12	9	10	11	4	1	2	3	8	5	6	7

FIGURE 6.4.3: Reduced-form block-Latin solutions to philatelic Sudoku puzzle S_4 blocked row-wise $\mathbf{B}_4^{(4,3)}$ (left panel) and column-wise $\mathbf{B}_4^{(3,4)}$ (right panel).

It is interesting to compare the generalized Kronecker-sum solutions (6.4.1) and (6.4.2) of size 12×12 to this philatelic Sudoku puzzle S_4 ($r = 8, c = 3, s = 12$) with the generalized Kronecker-sum solutions (6.3.3) and (6.3.5) of size 6×6 to the philatelic Sudoku puzzle S_3 ($r = 6, c = 3, s = 6$),

$$\mathbf{B}_3^{(2,3)} = \mathbf{E}_{2\times 2} \otimes \mathbf{K}_{3\times 3} + 3\mathbf{K}_{2\times 2} \otimes \mathbf{E}_{3\times 3} - 3\mathbf{E}_{2\times 2} \otimes \mathbf{E}_{3\times 3}, \tag{6.4.4}$$

$$\mathbf{B}_3^{(3,2)} = \mathbf{E}_{3\times 3} \otimes \mathbf{K}_{2\times 2} + 2\mathbf{K}_{3\times 3} \otimes \mathbf{E}_{2\times 2} - 2\mathbf{E}_{3\times 3} \otimes \mathbf{E}_{2\times 2}, \tag{6.4.5}$$

where the 2×2 one-step (backwards) circulant

$$\mathbf{K}_{2\times 2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \tag{6.4.6}$$

We find that the rank of the matrices in Figure 6.4.3 is equal to 6 and that a basis for the (column) null space of the matrix in the left-panel of Figure 6.4.3 is the set of columns of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.4.7}$$

6.5 Philatelic Sudoku puzzle S_5: stamps from the USA featuring "Beautiful Blooms"

FIGURE 6.5.1: USA 2007, *Scott* 4185v.

Our fifth philatelic Sudoku puzzle S_5 is based on a booklet of stamps (Figure 6.5.1) from the USA featuring "Beautiful Blooms", coded (from left-to-right in the top row) as follows: (1) coneflower, (2) tulip, (3) water lily, (4) poppy, (5) chrysanthemum, (6) orange Gerbera[30] daisy, (7) iris, (8) dahlia, (9) magnolia, (10) red Gerbera daisy.

To solve puzzle S_5 we block the stamps in pairs as follows:

$$a = (1,2), \quad b = (3,4), \quad c = (5,6), \quad d = (7,8), \quad e = (9,10). \tag{6.5.1}$$

[30]Gerbera is a genus of ornamental plants from the sunflower family (*Asteraceae*) named in honour of the German naturalist Traugott Gerber (1710–1743), a friend of Carl Linnaeus (1707–1778).

Then the stamps in Figure 6.5.1 may be represented by the matrix $\left(\begin{smallmatrix} a & b & c & d & e \\ \bar{e} & \bar{d} & b & a & c \end{smallmatrix}\right)$, where the super-bar reverses the numbers in the argument, e.g., $\bar{e} = (10, 9)$. Our solution to S_5 is easily found to be as shown in Figure 6.5.2:

$$\begin{pmatrix} a & b & c & d & e \\ \bar{e} & \bar{d} & b & a & c \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} & \bar{e} \\ e & d & \bar{b} & \bar{a} & \bar{c} \\ b & a & e & c & d \\ c & e & d & b & a \\ d & c & a & e & b \\ \bar{b} & \bar{a} & \bar{e} & \bar{c} & \bar{d} \\ \bar{c} & \bar{e} & \bar{d} & \bar{b} & \bar{a} \\ \bar{d} & \bar{c} & \bar{a} & \bar{e} & \bar{b} \end{pmatrix}$$

1	2	3	4	5	6	7	8	9	10
10	9	8	7	3	4	1	2	6	5
2	1	4	3	6	5	8	7	10	9
9	10	7	8	4	3	2	1	5	6
3	4	1	2	9	10	5	6	7	8
5	6	9	10	7	8	3	4	1	2
7	8	5	6	1	2	9	10	3	4
4	3	2	1	10	9	6	5	8	7
6	5	10	9	8	7	4	3	2	1
8	7	6	5	2	1	10	9	4	3

FIGURE 6.5.2: Solution to philatelic Sudoku puzzle S_5.

1	2	3	4	5	6	7	8	9	10
2	1	4	3	6	5	8	7	10	9
3	4	1	2	9	10	5	6	7	8
4	3	2	1	10	9	6	5	8	7
5	6	9	10	7	8	3	4	1	2
6	5	10	9	8	7	4	3	2	1
7	8	5	6	1	2	9	10	3	4
8	7	6	5	2	1	10	9	4	3
9	10	7	8	4	3	2	1	5	6
10	9	8	7	3	4	1	2	6	5

FIGURE 6.5.3: Reduced-form solution to philatelic Sudoku puzzle S_5.

If we rearrange the rows in the solution in Figure 6.5.2 so that the entries in column 1 are in ascending order then we obtain the reduced-form 5×5 block-Latin square shown in Figure 6.5.3, which allows the generalized Kronecker-sum representation

$$\mathbf{E}_{5\times 5} \otimes \mathbf{K}_{2\times 2} + 2\mathbf{L}_5 \otimes \mathbf{E}_{2\times 2} - 2\mathbf{E}_{5\times 5} \otimes \mathbf{E}_{2\times 2}, \tag{6.5.2}$$

where $\mathbf{E}$ has every element equal to 1, $\mathbf{K}_{2\times 2} = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, and the 5×5 reduced-form Latin square matrix

$$\mathbf{L}_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 4 & 2 & 1 & 3 \end{pmatrix}, \tag{6.5.3}$$

which we note is not a circulant, but is nonsingular (all 5×5 Latin square matrices are nonsingular). We find that the rank of the matrices in Figure 6.5.3 is equal to 7 and that a basis for its (column) null space is the set of columns of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{6.5.4}$$

6.6 Philatelic Sudoku puzzle S_6: stamps from the USA featuring "American Advances in Aviation"

FIGURE 6.6.1: USA 2007, *Scott* 3925v.

1	6	8	9	2	10	3	4	5	7
2	7	4	10	3	6	1	5	8	9
3	8	5	7	4	9	2	6	1	10
4	9	2	1	5	8	7	10	3	6
5	10	3	6	7	1	9	8	2	4
6	1	9	8	10	2	4	3	7	5
7	2	10	4	6	3	5	1	9	8
8	3	7	5	9	4	6	2	10	1
9	4	1	2	8	5	10	7	6	3
10	5	6	3	1	7	8	9	4	2

FIGURE 6.6.2: Solution to philatelic Sudoku puzzle S_6 (with givens in brown).

Our sixth (and last) philatelic Sudoku puzzle S_6 (Figure 6.6.1) depicts 10 different types of aircraft (coded column-wise): (1) Boeing 247, (2) Grumman F6F Hellcat, (3) Ercoupe 415, (4) Consolidated B-24 Liberator, (5) Beech 35 Bonanza, (6) Consolidated PBY Catalina, (7) Republic P-47 Thunderbolt, (8) Lockheed P-80 Shooting Star, (9) Boeing B-29 Superfortress, (10) Northrop YB-49 Flying Wing; see Figure 6.6.2 (top left panel in brown). We observe that the stamps are arranged in blocks of 10 (but not in sub-blocks of 5). We find the solution as shown in Figure 6.6.2 by completing first the 3 blocks in the top right by "trial and error". The remaining 5 blocks in the bottom part are then found by reversing the 5×1 sub-blocks, as shown schematically in the following matrix, with entries 5×1 vectors, e.g., $\mathbf{a}_1 = (1,2,3,4,5)'$,

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{e}_1 & \mathbf{e}_2 \\ \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{b}_2 & \mathbf{b}_1 & \mathbf{c}_2 & \mathbf{c}_1 & \mathbf{d}_2 & \mathbf{d}_1 & \mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix}, \tag{6.6.1}$$

and so each 10×2 submatrix, starting from the left, is 2×2 block-Latin. The matrix in Figure 6.6.2 is nonsingular.

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