Playing with Time and Playing in Time

Valentin Goranko
Stockholm University

Joint work with **Antti Kuusisto** and **Raine Rönnholm**

Lauri Hella 60 Fest
Murikanranta, July 6, 2018

10 sec trailer

Two main story lines:

1. Playing with Time: *game-theoretic semantics for branching time logic*

2. Playing in Time: *semantics with uniform time bounds on eventualities*

These meet naturally in the **finitely bounded semantics for the computation tree logic** CTL.

Outline of the talk

- Preliminaries: the computation tree logic CTL
- Game theoretic semantics for CTL
- CTL with finitely bounded semantics: $\mathrm{CTL}_{\mathbf{FB}}$
  - Semantics
  - Axiomatization
- Two versions of tableaux for $\mathrm{CTL}_{\mathbf{FB}}$: infinitary and finitary
- Decidability
- Concluding remarks

Preliminaries: the computation tree logic CTL

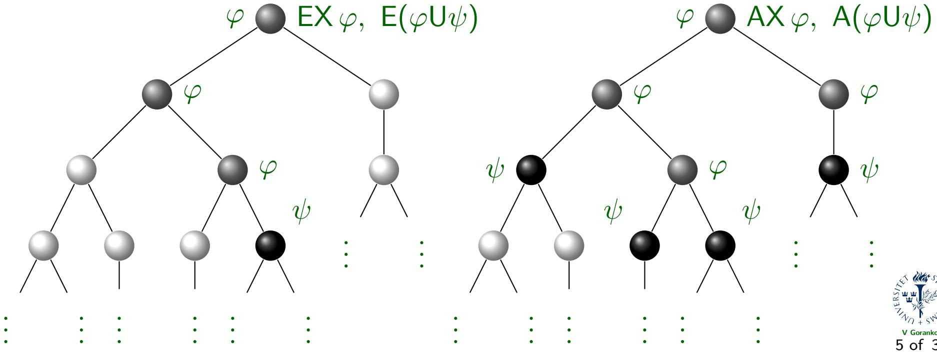
Preliminaries: the computation tree logic CTL

Formulae:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{EX}\,\varphi \mid \mathsf{E}(\varphi\,\mathsf{U}\,\varphi) \mid \mathsf{A}(\varphi\,\mathsf{U}\,\varphi)$$

Abbreviations: $\mathsf{AX}\,\varphi := \neg\mathsf{EX}\,\neg\varphi$,
$\mathsf{EF}\,\varphi := \mathsf{E}(\top\,\mathsf{U}\,\varphi)$, $\mathsf{AF}\,\varphi := \mathsf{A}(\top\,\mathsf{U}\,\varphi)$
$\mathsf{EG}\,\varphi := \neg\mathsf{AF}\,\neg\varphi$, $\mathsf{AG}\,\varphi := \neg\mathsf{EF}\,\neg\varphi$

Intuitive semantics of U:



Preliminaries: interpreted transition systems

An interpreted transition system (ITS):

$$\mathcal{M} = (\mathsf{S}, \mathsf{R}, \Phi, \mathsf{L}),$$

where

- S is a state space,
- $\mathsf{R} \subseteq \mathsf{S} \times \mathsf{S}$ is a transition relation,
- Φ a set of proposition symbols,
- $\mathsf{L} : \mathsf{S} \to \mathcal{P}(\Phi)$ is a state labelling function.

Preliminaries: formal compositional semantics of CTL

Truth of a CTL-formula φ at a state s in an ITS $\mathcal{M}$:

- $\mathcal{M}, s \models p$ iff $p \in \mathsf{L}(s)$
- $\mathcal{M}, s \models \neg\varphi$ iff $\mathcal{M}, s \not\models \varphi$
- $\mathcal{M}, s \models \varphi \vee \psi$ iff $\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$
- $\mathcal{M}, s \models \mathsf{EX}\,\varphi$ iff $\mathcal{M}, s' \models \varphi$ for some $s' \in \mathsf{S}$ such that $(s, s') \in \mathsf{R}$
- $\mathcal{M}, s \models \mathsf{E}(\varphi\,\mathsf{U}\,\psi)$ iff there is a path λ starting from s and $i \geq 0$ such that $\mathcal{M}, \lambda(i) \models \psi$ and $\mathcal{M}, \lambda(j) \models \varphi$ for every $j < i$
- $\mathcal{M}, s \models \mathsf{A}(\varphi\,\mathsf{U}\,\psi)$ iff for every path λ starting from s, there is $i \geq 0$ such that $\mathcal{M}, \lambda(i) \models \psi$ and $\mathcal{M}, \lambda(j) \models \varphi$ for every $j < i$

Derived clauses:

- $\mathcal{M}, s \models \mathsf{EG}\,\psi$ iff there is a path λ starting from s such that $\mathcal{M}, \lambda(i) \models \psi$ for every $i \geq 0$
- $\mathcal{M}, s \models \mathsf{AG}\,\psi$ iff for every path λ starting from s, $\mathcal{M}, \lambda(i) \models \psi$ for every $i \geq 0$.

Fixpoint definitions of the CTL operators in the standard semantics

Operators on formulae, where $\mathsf{Q} \in \{\mathsf{E}, \mathsf{A}\}$:

$$\mathbf{U}_{\mathsf{Q};\psi,\theta}(\varphi) := \theta \vee (\psi \wedge \mathsf{QX}\,\varphi); \quad \mathbf{G}_{\mathsf{Q};\theta}(\varphi) := \theta \wedge \mathsf{QX}\,\varphi.$$

Fixpoint characterisations in the standard semantics:

- $\mathsf{Q}(\psi\,\mathsf{U}\,\theta)$ is the least fixpoint of the operator $\mathbf{U}_{\mathsf{Q};\psi,\theta}$
  i.e., $\mathsf{E}(\psi\,\mathsf{U}\,\theta) \equiv \mu Z.\mathbf{U}_{\mathsf{E};\psi,\theta}(Z)$, $\mathsf{A}(\psi\,\mathsf{U}\,\theta) \equiv \mu Z.\mathbf{U}_{\mathsf{A};\psi,\theta}(Z)$.
- $\mathsf{QG}\,\theta$ is the greatest fixpoint of the operator $\mathbf{G}_{\mathsf{Q};\theta}$
  i.e., $\mathsf{EG}\,\theta \equiv \nu Z.\mathbf{G}_{\mathsf{E};\theta}(Z)$, $\mathsf{AG}\,\theta \equiv \nu Z.\mathbf{G}_{\mathsf{A};\theta}(Z)$

We define inductively on $n \in \mathbb{N}$ the iterations of these operators:

- $\mathbf{U}^0_{\mathsf{Q}}(\psi,\theta) := \theta$; $\mathbf{U}^{n+1}_{\mathsf{Q}}(\psi,\theta) := \mathbf{U}_{\mathsf{Q};\psi,\theta}(\mathbf{U}^n_{\mathsf{Q}}(\psi,\theta))$.
- $\mathbf{G}^0_{\mathsf{Q}}(\theta) := \theta$; $\mathbf{G}^{n+1}_{\mathsf{Q}}(\theta) := \mathbf{G}_{\mathsf{Q};\theta}(\mathbf{G}^n_{\mathsf{Q}}(\theta))$

Complete axiomatic system for CTL

The first complete axiomatic system for CTL was proposed by Emerson and Halpern in 1982. Here is a streamlined version:

Axiom schemata:

Enough classical tautologies.

(K_X) $\mathsf{AX}\,(\varphi \to \psi) \to (\mathsf{AX}\,\varphi \to \mathsf{AX}\,\psi)$

(D_X) $\mathsf{EX}\,\top$

(FP_{EU}) $\mathsf{E}(\varphi\,\mathsf{U}\,\psi) \leftrightarrow (\psi \vee (\varphi \wedge \mathsf{EX}\,\mathsf{E}(\varphi\,\mathsf{U}\,\psi)))$
($\mathsf{E}(\psi\,\mathsf{U}\,\theta)$ is a fixpoint of the operator $\mathbf{U}_{\mathsf{E};\psi,\theta}$)

(FP_{AU}) $\mathsf{A}(\varphi\,\mathsf{U}\,\psi) \leftrightarrow (\psi \vee (\varphi \wedge \mathsf{AX}\,\mathsf{A}(\varphi\,\mathsf{U}\,\psi)))$
($\mathsf{A}(\psi\,\mathsf{U}\,\theta)$ is a fixpoint of the operator $\mathbf{U}_{\mathsf{A};\psi,\theta}$)

(LFP_{EU}) $\mathsf{AG}\,((\psi \vee (\varphi \wedge \mathsf{EX}\,\chi)) \to \chi) \to (\mathsf{E}(\varphi\,\mathsf{U}\,\psi) \to \chi)$
($\mathsf{E}(\psi\,\mathsf{U}\,\theta)$ is a least pre-fixpoint of the operator $\mathbf{U}_{\mathsf{E};\psi,\theta}$)

(LFP_{AU}) $\mathsf{AG}\,((\psi \vee (\varphi \wedge \mathsf{AX}\,\chi)) \to \chi) \to (\mathsf{A}(\varphi\,\mathsf{U}\,\psi) \to \chi)$
($\mathsf{A}(\psi\,\mathsf{U}\,\theta)$ is a least pre-fixpoint of the operator $\mathbf{U}_{\mathsf{A};\psi,\theta}$)

Rules: **Modus ponens** and **Necessitation** $\mathrm{NEC}_{\mathsf{AG}}$: $\vdash \varphi$ implies $\vdash \mathsf{AG}\,\varphi$.

Game-theoretic semantics for CTL

Game-theoretic semantics for CTL

In *game-theoretic semantics* (GTS), truth of a formula φ is determined in a formal dispute, called *evaluation game*, between two players:

Eloise, who is trying to *verify* φ, and Abelard, who is trying to *falsify* it.

GTS defines truth of φ as
existence of a winning strategy for Eloise in the evaluation game for φ.

The (unbounded) evaluation game for CTL

Let $\mathcal{M} = (\mathsf{S}, \mathsf{R}, \Phi, \mathsf{L})$ be an ITS, $s_{in} \in \mathsf{S}$ and φ a CTL-formula.

Brief description of the *(unbounded) evaluation game* $\mathcal{G}(\mathcal{M}, s_{in}, \varphi)$

A position of the game is a tuple $(\mathbf{P}, s, \psi)$,
where $\mathbf{P} \in \{\text{Abelard}, \text{Eloise}\}$, $s \in \mathsf{S}$ and ψ is a subformula of φ.

The game $\mathcal{G}$ begins from the initial position $(\text{Eloise}, s_{in}, \varphi)$ and proceeds according to specific rules for each logical connective.

For the temporal connectives $\mathsf{E\,U}$ and $\mathsf{A\,U}$ the game $\mathcal{G}$ invokes embedded subgames that consist in an unbounded number of steps.

Rules for the evaluation game

1. A position $(\mathbf{P}, s, p)$, where $p \in \Phi$ is an ending position.
   If $p \in \mathsf{L}(s)$, then $\mathbf{P}$ wins the evaluation game.
   Else the opposing player $\overline{\mathbf{P}}$ wins.

2. In $(\mathbf{P}, s, \neg\psi)$ the game moves to the next position $(\overline{\mathbf{P}}, s, \psi)$.

3. In $(\mathbf{P}, s, \psi \vee \theta)$ the player $\mathbf{P}$ chooses the next position: $(\mathbf{P}, s, \psi)$ or $(\mathbf{P}, s, \theta)$.

4. In $(\mathbf{P}, s, \mathsf{EX}\,\psi)$ the player $\mathbf{P}$ may choose any state s' such that $(s, s') \in \mathsf{R}$ and the next position is $(\mathbf{P}, s', \psi)$.

The rules for the formulae $\mathsf{E}(\psi\,\mathsf{U}\,\theta)$ and $\mathsf{A}(\psi\,\mathsf{U}\,\theta)$, send the players to play an embedded subgame.

It ends with an exit position, from which the evaluation game resumes.

The embedded subgame $\mathbf{G}$

$\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{L}, s_0, \psi_{\mathbf{V}}, \psi_{\overline{\mathbf{V}}})$, where

$\mathbf{V}, \mathbf{L} \in \{Abelard, Eloise\}$, s_0 is a state, and $\psi_{\mathbf{V}}$ and $\psi_{\overline{\mathbf{V}}}$ are formulae.

$\mathbf{V}$ is the verifier in $\mathbf{G}$, and $\mathbf{L}$ the leader. These may be the same.

$\overline{\mathbf{V}}$ and $\overline{\mathbf{L}}$ denote the opponents of $\mathbf{V}$ and $\mathbf{L}$, respectively.

$\mathbf{G}$ starts from the initial state s_0 and proceeds from any state s according to the following rules until an exit position is reached.

i) $\mathbf{V}$ may end the game at the exit position $(\mathbf{V}, s, \psi_{\mathbf{V}})$.

ii) $\overline{\mathbf{V}}$ may end the game at the exit position $(\mathbf{V}, s, \psi_{\overline{\mathbf{V}}})$.

iii) $\mathbf{L}$ may select any state s' such that $(s, s') \in \mathrm{R}$. Then $\mathbf{G}$ continues from s'.

If the embedded game $\mathbf{G}$ continues an infinite number of rounds, then the verifier $\mathbf{V}$ loses the entire evaluation game.

The rest of the rules for the evaluation game are as follows:

5. In $(\mathbf{P}, s, \mathrm{E}(\psi \,\mathrm{U}\, \theta))$ the game is continued from the exit position of $\mathbf{g}(\mathbf{P}, \mathbf{P}, s, \theta, \psi)$.
6. In $(\mathbf{P}, s, \mathrm{A}(\psi \,\mathrm{U}\, \theta))$ the game is continued from the exit position of $\mathbf{g}(\mathbf{P}, \overline{\mathbf{P}}, s, \theta, \psi)$.

The (unbounded) game-theoretic semantics for CTL

Unbounded game-theoretic semantics for CTL :

$\mathcal{M}, s \models_{GTS} \varphi$ *iff Eloise has a winning strategy in* $\mathcal{G}(\mathcal{M}, s, \varphi)$.

Theorem

The unbounded GTS for CTL is equivalent to the standard, compositional semantics of CTL.

The unbounded evaluation games are determined, but possibly infinite.

Can we make them finite?

Yes, by imposing time bounds.

The (ordinal) bounded game-theoretic semantics for CTL

Evaluation games can be modified by assigning ordinal time limits to the embedded subgames. That leads to ordinal bounded evaluation games.

The time limit is an ordinal announced by Verifier at the beginning of the embedded subgame and Verifier has to decrease it after every transition.

Since ordinals are well-founded, the evaluation game is guaranteed to end in a finite number of moves—even in infinite models.

Thus, the (ordinal) bounded GTS is obtained.

Theorem

The ordinal bounded GTS for CTL is equivalent to the unbounded GTS.

I will now focus on evaluation games with finite time limits.

These define the finitely bounded GTS for CTL.

CTL with finitely bounded semantics

Finitely bounded compositional semantics for CTL

The finitely bounded GTS ($\mathrm{GTS_{fb}}$) modifies the truth conditions of AU and EU by imposing a uniform bound on the number of transition steps needed to fulfil a given eventuality:

(AU$_{\mathrm{fb}}$) $\mathcal{M}, s \models_{\mathrm{fb}} \mathsf{A}(\varphi \,\mathsf{U}\, \psi)$ iff there is $n \in \mathbb{N}$ such that for every history λ starting from s, there is $i \leq n$ such that $\mathcal{M}, \lambda(i) \models_{\mathrm{fb}} \psi$ and $\mathcal{M}, \lambda(j) \models_{\mathrm{fb}} \varphi$ for every $j < i$.

(EU$_{\mathrm{fb}}$) $\mathcal{M}, s \models_{\mathrm{fb}} \mathsf{E}(\varphi \,\mathsf{U}\, \psi)$ iff there is $n \in \mathbb{N}$, a history λ starting from s and $i \leq n$ such that $\mathcal{M}, \lambda(i) \models_{\mathrm{fb}} \psi$ and $\mathcal{M}, \lambda(j) \models_{\mathrm{fb}} \varphi$ for every $j < i$.

(EU$_{\mathrm{fb}}$) is in fact equivalent to the standard truth definition of EU.

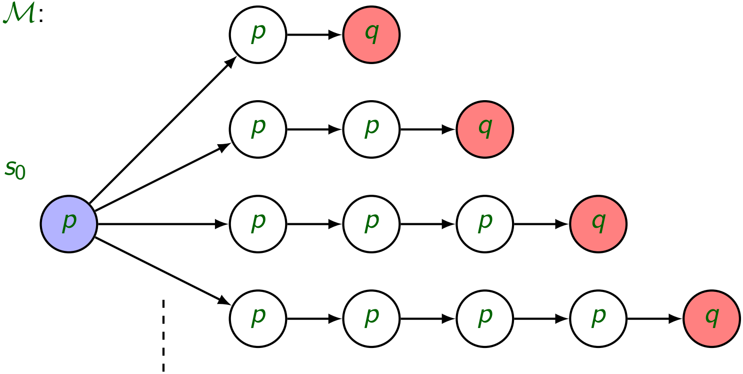
The derived clause for AG is equivalent to the standard one. For EG:

(EG$_{\mathrm{fb}}$) $\mathcal{M}, s \models_{\mathrm{fb}} \mathsf{EG}\,\varphi$ iff for every $n \in \mathbb{N}$, there is a history λ_n starting from s such that $\mathcal{M}, \lambda_n(i) \models_{\mathrm{fb}} \varphi$ for every $i \leq n$. (Note that the history λ_n depends on n.)

By replacing the truth condition for AU and EG with the ones above, we obtain CTL with finitely bounded semantics, denoted by $\mathrm{CTL}_{\mathbf{FB}}$.

Example

$\mathcal{M}$:



$\mathcal{M}, s_0 \models \mathsf{A}(p \, \mathsf{U} \, q)$ but $\mathcal{M}, s_0 \not\models_{\mathrm{fb}} \mathsf{A}(p \, \mathsf{U} \, q)$

In terms of the GTS: Eloise can win $\mathcal{G}(\mathcal{M}, s_0, p \, \mathsf{U} \, q)$ in the unbounded evaluation game, or in the ordinal-bounded one, but not in the bounded version with finite time limits.

Respectively, $\mathcal{M}, s_0 \not\models \mathsf{EG} \, p$ but $\mathcal{M}, s_0 \models_{\mathrm{fb}} \mathsf{EG} \, p$.

Some properties of $\mathrm{CTL}_{\mathsf{FB}}$

1. $\mathrm{CTL}_{\mathsf{FB}} = \mathrm{CTL}$ on all *image finite models* i.e., truth of CTL-formulae on these is independent of which semantics is used.

2. $\mathrm{CTL} \neq \mathrm{CTL}_{\mathsf{FB}}$ on models that have infinite branchings.
   In particular, the fixed point properties of the operators F and G fail since the implications $\mathsf{EG}\, p \to (p \wedge \mathsf{EX}\, \mathsf{EG}\, p)$ and (dually) $(p \vee \mathsf{AX}\, \mathsf{AF}\, p) \to \mathsf{AF}\, p$ are valid in CTL but not in $\mathrm{CTL}_{\mathsf{FB}}$.

3. Consequently, $\mathrm{CTL}_{\mathsf{FB}}$ does not have the finite model property, as these implications cannot fail in (image-)finite models.

4. Therefore, the validities of $\mathrm{CTL}_{\mathsf{FB}}$ are properly included in the validities of CTL.

   Indeed, every non-validity of CTL is falsified in a finite model and thus, by fact 1, it is a non-validity of $\mathrm{CTL}_{\mathsf{FB}}$, too.

The questions of axiomatisation and decidability of $\mathrm{CTL}_{\mathsf{FB}}$ arise.

Axiomatic system for $\mathrm{CTL}_{\mathbf{FB}}$

Recalling the fixpoint definitions of CTL operators

Operators on formulae, where $\mathsf{Q} \in \{\mathsf{E}, \mathsf{A}\}$:

$$\mathbf{G}_{\mathsf{Q};\theta}(\varphi) := \theta \wedge \mathsf{QX}\,\varphi; \quad \mathbf{U}_{\mathsf{Q};\psi,\theta}(\varphi) := \theta \vee (\psi \wedge \mathsf{QX}\,\varphi).$$

The iterations of these operators defined recursively on $n \in \mathbb{N}$:

- $\mathbf{G}^0_{\mathsf{Q}}(\theta) := \theta$; $\mathbf{G}^{n+1}_{\mathsf{Q}}(\theta) := \mathbf{G}_{\mathsf{Q};\theta}(\mathbf{G}^n_{\mathsf{Q}}(\theta))$
- $\mathbf{U}^0_{\mathsf{Q}}(\psi,\theta) := \theta$; $\mathbf{U}^{n+1}_{\mathsf{Q}}(\psi,\theta) := \mathbf{U}_{\mathsf{Q};\psi,\theta}(\mathbf{U}^n_{\mathsf{Q}}(\psi,\theta))$.

Axiomatic system for $\mathrm{CTL}_{\mathbf{FB}}$: axioms from CTL

Axiom schemata:

Enough classical tautologies.

$(\mathsf{K_X})$ $\mathsf{AX}(\varphi \to \psi) \to (\mathsf{AX}\,\varphi \to \mathsf{AX}\,\psi)$

$(\mathsf{D_X})$ $\mathsf{EX}\,\top$

$(\mathsf{FP_{AG}})$ $(\varphi \land \mathsf{AX}\,\mathsf{AG}\,\varphi) \leftrightarrow \mathsf{AG}\,\varphi$
($\mathsf{AG}\,\varphi$ is a fixed point of the operator $\mathbf{G}_{\mathsf{A};\varphi}$)

$(\mathsf{G\text{-}PostFP_{AG}})$ $\mathsf{AG}(\psi \to (\varphi \land \mathsf{AX}\,\psi)) \to (\psi \to \mathsf{AG}\,\varphi)$
($\mathsf{AG}\,\varphi$ is the greatest post-fixed point of $\mathbf{G}_{\mathsf{A};\varphi}$)

$(\mathsf{FP_{EU}})$ $\mathsf{E}(\varphi\,\mathsf{U}\,\psi) \leftrightarrow (\psi \lor (\varphi \land \mathsf{EX}\,\mathsf{E}(\varphi\,\mathsf{U}\,\psi)))$
($\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$ is a fixed point of the operator $\mathbf{U}_{\mathsf{E};\varphi,\psi}$)

$(\mathsf{L\text{-}PreFP_{EU}})$ $\mathsf{AG}((\psi \lor (\varphi \land \mathsf{EX}\,\chi)) \to \chi) \to (\mathsf{E}(\varphi\,\mathsf{U}\,\psi) \to \chi)$
($\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$ is the least pre-fixed point of $\mathbf{U}_{\mathsf{E};\varphi,\psi}$)

Axiomatic system for $\mathrm{CTL}_{\mathbf{FB}}$: new axioms

(PreFP$_{\mathsf{EG}}$) $(\varphi \wedge \mathsf{EX}\,\mathsf{EG}\,\varphi) \to \mathsf{EG}\,\varphi$
(EG φ is a pre-fixed point of the operator $\mathbf{G}_{\mathsf{E};\varphi}$)

(UB-PostFP$_{\mathsf{EG}}$) $\mathsf{AG}\,(\psi \to (\varphi \wedge \mathsf{EX}\,\psi)) \to (\psi \to \mathsf{EG}\,\varphi)$
(EG φ is an upper bound for all post-fixed points of $\mathbf{G}_{\mathsf{E};\varphi}$)

(PostFP$_{\mathsf{AU}}$) $\mathsf{A}(\varphi\,\mathsf{U}\,\psi) \to (\psi \vee (\varphi \wedge \mathsf{AX}\,\mathsf{A}(\varphi\,\mathsf{U}\,\psi)))$
($\mathsf{A}(\varphi\,\mathsf{U}\,\psi)$ is a post-fixed point of $\mathbf{U}_{\mathsf{A};\varphi,\psi}$)

(LB-PreFP$_{\mathsf{AU}}$) $\mathsf{AG}\,((\psi \vee (\varphi \wedge \mathsf{AX}\,\chi)) \to \chi) \to (\mathsf{A}(\varphi\,\mathsf{U}\,\psi) \to \chi)$
($\mathsf{A}(\varphi\,\mathsf{U}\,\psi)$ is a lower bound for all pre-fixed points of $\mathbf{U}_{\mathsf{A};\varphi,\psi}$)

Additional infinite schemes of axioms (replacing in $\mathrm{CTL}_{\mathbf{FB}}$ the missing directions of the standard CTL fixed-point equivalences), for $\mathsf{Q} \in \{\mathsf{E}, \mathsf{A}\}$:

(EG$^{\infty}$) $\mathsf{EG}\,\varphi \to \mathbf{G}^{n}_{\mathsf{E}}(\varphi)$, for every $n \in \mathbb{N}$.

(AU$^{\infty}$) $\mathbf{U}^{n}_{\mathsf{A}}(\varphi, \psi) \to \mathsf{A}(\varphi\,\mathsf{U}\,\psi)$, for every $n \in \mathbb{N}$.

Axiomatic system for $\mathrm{CTL}_{\mathbf{FB}}$: rules

Standard rules:

Modus ponens and **Necessitation** $\mathsf{NEC}_{\mathsf{AG}}$: $\vdash \varphi$ implies $\vdash \mathsf{AG}\,\varphi$

Infinitary rules:

EG-**Accumulation:**

$$\frac{\vdash \theta \to \mathbf{G}^n_{\mathsf{E}}(\varphi), \text{ for every } n \in \mathbb{N}}{\vdash \theta \to \mathsf{EG}\,\varphi}$$

$\mathsf{A\,U}$-**Accumulation:**

$$\frac{\vdash \mathbf{U}^n_{\mathsf{A}}(\varphi, \psi) \to \theta, \text{ for every } n \in \mathbb{N}}{\vdash \mathsf{A}(\varphi\,\mathsf{U}\,\psi) \to \theta}$$

Axiomatic system for $\mathrm{CTL}_{\mathbf{FB}}$: results

Theorem: $\mathbf{Ax}_{\mathrm{CTL}_{\mathbf{FB}}}$ is sound and complete.

Proposition: $\mathrm{CTL}_{\mathbf{FB}}$ is not finitely axiomatizable.

Open question: Are the infinitary rules redundant?

Tableaux for $\mathrm{CTL}_{\mathbf{FB}}$

Types and components of formulae in $\mathrm{CTL}_{\mathbf{FB}}$

successor formula	successor component
$\mathsf{EX}\,\varphi$ (existential)	φ
$\mathsf{AX}\,\varphi$ (universal)	φ
$\neg\mathsf{AX}\,\varphi$ (existential)	$\neg\varphi$
$\neg\mathsf{EX}\,\varphi$ (universal)	$\neg\varphi$

conjunctive formula	conjunctive components	disjunctive formula	disjunctive components
$\neg\neg\varphi$	φ		
$\varphi \wedge \psi$	$\varphi,\ \psi$	$\neg(\varphi \wedge \psi)$	$\neg\varphi,\ \neg\psi$
$\mathsf{AG}\,\varphi$	$\{\varphi, \mathsf{AX}\,\mathsf{AG}\,\varphi\}$	$\neg\mathsf{AG}\,\varphi$	$\{\neg\mathbf{G}^n_{\mathsf{A}}(\varphi)\}_{n\in\mathbb{N}}$
$\mathsf{EG}\,\varphi$	$\{\mathbf{G}^n_{\mathsf{E}}(\varphi)\}_{n\in\mathbb{N}}$	$\neg\mathsf{EG}\,\varphi$	$\{\neg\mathbf{G}^n_{\mathsf{E}}(\varphi)\}_{n\in\mathbb{N}}$
$\neg\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$	$\{\neg\psi, \neg\varphi \vee \neg\mathsf{EX}\,\mathsf{E}(\varphi\,\mathsf{U}\,\psi)\}$	$\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$	$\{\mathbf{U}^n_{\mathsf{E}}(\varphi, \psi)\}_{n\in\mathbb{N}}$
$\neg\mathsf{A}(\varphi\,\mathsf{U}\,\psi)$	$\{\neg\mathbf{U}^n_{\mathsf{A}}(\varphi, \psi)\}_{n\in\mathbb{N}}$	$\mathsf{A}(\varphi\,\mathsf{U}\,\psi)$	$\{\mathbf{U}^n_{\mathsf{A}}(\varphi, \psi)\}_{n\in\mathbb{N}}$

Closure set *ecl*(η) of a formula η:
the least set containing η and closed under taking components.

Infinitary tableaux for $\mathrm{CTL}_{\mathbf{FB}}$ in a nutshell

- Built incrementally from an initial formula η, by alternating construction of pre-states and states.
- Every state labelled with a 'fully expanded' subset of $ecl(\eta)$.
- Three phases: construction phase, pre-state elimination, and state elimination phase.
- New 'successor states' created in the construction phase, to 'satisfy' existential successor formulae in the label of the current state.
- No states with repeating labels created, but looping back to existing states with the needed labels.
- States that do not have the necessary successors are eliminated.
- The constructed tableau may be infinite, and the elimination phase may go on in a *transfinite number of steps*, until stabilisation.

The final tableau is obtained when the elimination phase is completed.

It is open if at least one state has η in its label, otherwise closed.

An open tableau produces a satisfying Hintikka structure, from which a satisfying model can be constructed.

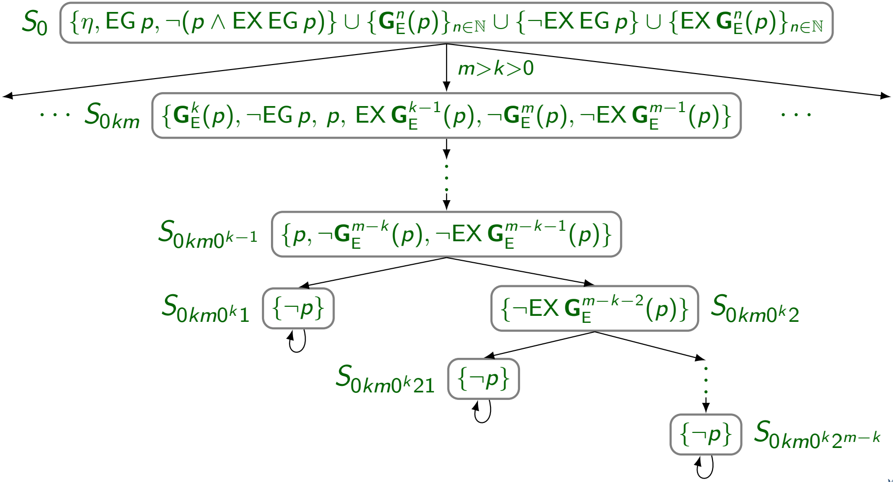
Infinitary tableaux for $\mathrm{CTL}_{\textbf{FB}}$: soundness and completeness

Theorem

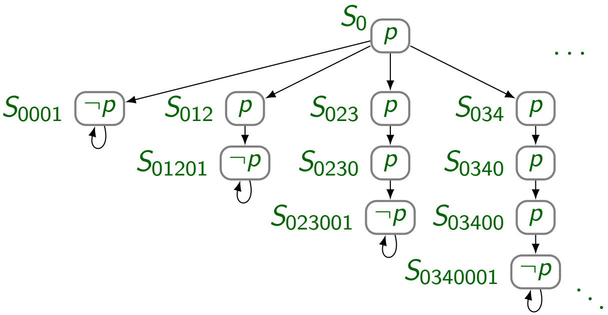
The infinitary tableau for $\eta \in \mathrm{CTL}_{\textbf{FB}}$ is sound and complete: for any formula $\eta \in \mathrm{CTL}_{\textbf{FB}}$, the final tableau $\mathcal{T}^{\eta}$ is open iff η is satisfiable.

Infinitary tableaux for $\mathrm{CTL}_{\mathbf{FB}}$: example

Infinitary tableau for $\eta = \mathsf{EG}\,p \wedge \neg(p \wedge \mathsf{EX}\,\mathsf{EG}\,p)$:



A model satisfying $\eta = \mathsf{EG}\, p \wedge \neg(p \wedge \mathsf{EX}\,\mathsf{EG}\, p)$



Recall: $\mathcal{M}, s \models_{\mathrm{fb}} \mathsf{EG}\,\varphi$ iff

for every $n \in \mathbb{N}$, there is a history λ_n starting from s such that $\mathcal{M}, \lambda_n(i) \models_{\mathrm{fb}} \varphi$ for every $i \leq n$.

Towards finitary tableaux: the extended language $\mathrm{CTL}^{par}_{\mathbf{FB}}$

We add a set of new symbols $\{\mathbf{n}_i \mid i \in \mathbb{N}^+\}$, called iteration parameters, replacing natural numbers n_i in formulae of type $\mathbf{G}^{n_i}_{\mathsf{Q}}(\varphi)$ and $\mathbf{U}^{n_i}_{\mathsf{Q}}(\varphi, \psi)$.

The resulting extended language: $\mathrm{CTL}^{par}_{\mathbf{FB}}$

NB: $\mathbf{G}^{\mathbf{n}_i}_{\mathsf{Q}}(\varphi), \mathbf{U}^{\mathbf{n}_i}_{\mathsf{Q}}(\varphi, \psi) \in \mathrm{CTL}^{par}_{\mathbf{FB}}$ *are not abbreviations*.
They are treated as actual formulae, only for the tableaux construction.

The iteration parameter $\mathbf{n}_i$ is just a symbol. It has no concrete value.

Intuitively, $\mathbf{n}_i$ takes an "arbitrarily large" but finite and fixed value which represents the number of iterations.

The index $i \in \mathbb{N}^+$ indicates *when* the "value" of $\mathbf{n}_i$ has been fixed (with respect to the other iteration parameters).

Finitary tableaux for $\mathrm{CTL}_{\mathbf{FB}}$

Components of some formulae in $\mathrm{CTL}_{\mathbf{FB}}$ are now re-defined as follows:

formulae	conjunctive components
$\mathsf{AG}\,\varphi$	$\{\varphi, \mathsf{AX}\,\mathsf{AG}\,\varphi\}$
$\mathsf{EG}\,\varphi$, $\mathbf{G}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi)$	$\{\varphi, \mathsf{EX}\,\mathbf{G}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi)\}$
$\neg\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$	$\{\neg\psi, \neg\varphi \vee \neg\mathsf{EX}\,\mathbf{U}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi,\psi)\}$
$\neg\mathsf{A}(\varphi\,\mathsf{U}\,\psi), \neg\mathbf{U}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi,\psi)$	$\{\neg\psi, \neg\varphi \vee \neg\mathsf{AX}\,\mathbf{U}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi,\psi)\}$

formulae	disjunctive component
$\neg\mathsf{AG}\,\varphi$, $\neg\mathbf{G}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi)$	$\{\neg\varphi, \neg\mathsf{AX}\,\mathbf{G}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi)\}$
$\neg\mathsf{EG}\,\varphi$, $\neg\mathbf{G}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi)$	$\{\neg\varphi, \neg\mathsf{EX}\,\mathbf{G}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi)\}$
$\mathsf{E}(\varphi\,\mathsf{U}\,\psi)$, $\mathbf{U}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi,\psi)$	$\{\psi, \varphi \wedge \mathsf{EX}\,\mathbf{U}_{\mathsf{E}}^{\mathbf{n}_i}(\varphi,\psi)\}$
$\mathsf{A}(\varphi\,\mathsf{U}\,\psi)$, $\mathbf{U}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi,\psi)$	$\{\psi, \varphi \wedge \mathsf{AX}\,\mathbf{U}_{\mathsf{A}}^{\mathbf{n}_i}(\varphi,\psi)\}$

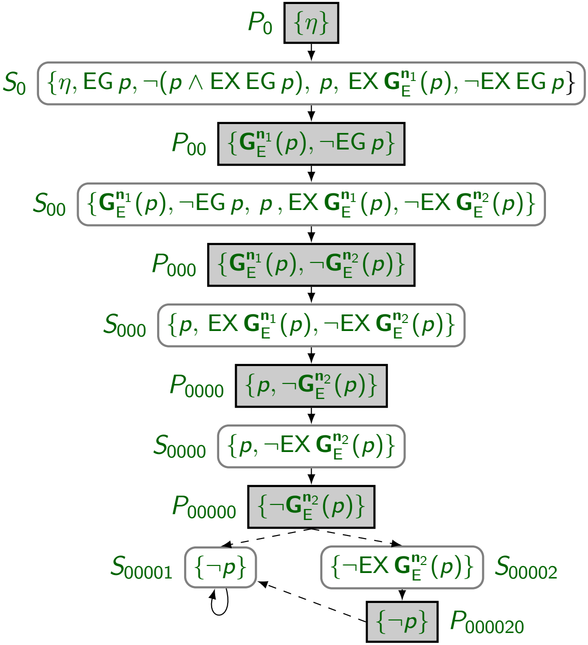
Closure sets of formulae and full expansions are defined as before.

New parameters can be introduced in the full expansions.

The tableaux building and state elimination phases are suitably modified.

The tableaux are now always finite.

Finitary tableaux for $\eta = \mathrm{EG}\, p \wedge \neg(p \wedge \mathrm{EX}\,\mathrm{EG}\, p)$



Results

Theorem
The infinitary tableau for any formula $\eta \in \mathrm{CTL}_{\boldsymbol{FB}}$ is open if and only if the finitary tableau for η is open.

Corollary
The finitary tableau for $\mathrm{CTL}_{\boldsymbol{FB}}$ is sound and complete.

Theorem
The complexity of running the finitary tableau for $\mathrm{CTL}_{\boldsymbol{FB}}$ is $\mathrm{EXPTIME}$-complete.

Corollary
The satisfiability problem of $\mathrm{CTL}_{\boldsymbol{FB}}$ is decidable and $\mathrm{EXPTIME}$-complete.

Summary and concluding remarks

The motivation for the logic $\mathrm{CTL}_{\mathbf{FB}}$ was two-fold:

– natural game-theoretic semantics,

– uniform boundedness of the time limit for satisfaction of eventualities across all branches.

Both apply beyond CTL and also produce respective finitely bounded versions of other logics, e.g. CTL^*, the modal mu-calculus, and ATL.

$\mathrm{CTL}_{\mathbf{FB}}$ has some special features, incl. the lack of finite model property.

That, in particular, requires an infinitary Hilbert-style axiomatization.

Likewise, the natural tableau for $\mathrm{CTL}_{\mathbf{FB}}$ is infinitary, but can be made finitary by symbolic treatment of infinite bundles of similar branches, thus providing a decision method for the satisfiability in $\mathrm{CTL}_{\mathbf{FB}}$.

The end (almost)

HAPPY 60th ANNIVERSARY, LAURI!