Expressivity within second-order transitive-closure logic

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Transitive closure

The transitive closure $\mathrm{TC}(R)$ of a binary relation $R \subseteq A \times A$ is defined as follows

$$\mathrm{TC}(R) := \{(a, b) \in A \times A \mid \exists n > 0 \text{ and } e_0, \ldots, e_n \in A \text{ such that } a = e_0,\ b = e_n, \text{ and } (e_i, e_{i+1}) \in R \text{ for all } i < n\}.$$

In our setting A is set of tuples $(a_1, \ldots a_n)$, where each a_i is either an *element* or a *relation* over some domain D.

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Example

Let $G = (V, E)$ be an undirected graph. Then $(a, b) \in \mathrm{TC}(E)$ if a and b are in the same component of G, or equivalently, if there is a path from a to b in G.

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A graph $G = (V, E)$ has a Hamiltonian cycle if the following holds:

1. There is a relation $\mathcal{R}$ such that
$$(Z, z, Z', z') \in \mathcal{R} \quad \text{iff} \quad Z' = Z \cup \{z'\}, z' \notin Z \text{ and } (z, z') \in E.$$
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Logics with transitive closure operator

First-order transitive closure logic $\mathrm{FO(TC)}$:

$$\varphi ::= x = y \mid X(x_1, \dots, x_k) \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists x \varphi \mid [\mathrm{TC}_{\vec{x},\vec{x'}}\varphi](\vec{y}, \vec{y'}),$$

where $\vec{x}$, $\vec{x'}$,$\vec{y}$, and $\vec{y'}$ are tuples of first-order variables of the same length.

Semantics for the TC operator:

$$\mathfrak{A} \models_s [\mathrm{TC}_{\vec{x},\vec{x'}}\varphi](\vec{y}, \vec{y'}) \text{ iff } \big(s(\vec{y}), s(\vec{y'})\big) \in \mathrm{TC}(\{(\vec{a}, \vec{a'}) \mid \mathfrak{A} \models_{s(\vec{x}\mapsto\vec{a}, \vec{x'}\mapsto\vec{a'})} \varphi\})$$

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The sentence

$$\forall x \forall y [\mathrm{TC}_{z,z'} E(z, z')](x, y)$$

expresses connectivity of graphs (V, E).

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The Härtig quantifier

$$\mathfrak{A} \models_s \mathrm{H}xy(\varphi(x), \psi(y)) \Leftrightarrow \text{the sets } \{a \in A \mid \mathfrak{A} \models_{s(x \mapsto a)} \varphi(x)\} \text{ and } \{b \in A \mid \mathfrak{A} \models_{s(y \mapsto b)} \psi(y)\} \text{ have the same cardinality}$$

Example (The Härtig quantifier can be expressed in MSO(TC).)

Let $\psi_{\mathrm{decrement}}$ denote an FO-formula expressing that $s(X') = s(X) \setminus \{a\}$ and $s(Y') = s(Y) \setminus \{b\}$ for some a and b. Define

$$\psi_{\mathrm{ec}} := \exists X_{\emptyset}\Big((\forall x \neg X_{\emptyset}(x)) \wedge [\mathrm{TC}_{X,Y,X',Y'}\psi_{\mathrm{decrement}}](Z, Z', X_{\emptyset}, X_{\emptyset})\Big).$$

Now ψ_{ec} holds under s if and only if the cardinalities of $s(Z)$ and $s(Z')$ are the same. Therefore $\mathrm{H}xy(\varphi(x), \psi(y))$ is equivalent with the formula

$$\exists Z \exists Z' (\forall x(\varphi(x) \leftrightarrow Z(x)) \wedge \forall y(\psi(y) \leftrightarrow Z'(y)) \wedge \psi_{\mathrm{ec}}).$$

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In the language of $\mathrm{MSO(TC)}$ this can be written as follows:

$$\exists XYxy(X(x) \wedge \forall z(z \neq x \rightarrow \neg X(x)) \wedge \forall z(Y(z)) \wedge E(y, x) \wedge [\mathrm{TC}_{Z,z,Z',z'}\varphi](X, x, Y, y))$$

where $\varphi := \neg Z(z') \wedge \forall x(Z'(x) \leftrightarrow (Z(x) \vee z' = x)) \wedge E(z, z')$.

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Descriptive complexity

Theorem (Harel and Peleg 84)

$\mathrm{SO(TC)}$ *captures polynomial space* PSPACE.

Theorem (Immerman 87)

- *On finite ordered structures, first-order transitive-closure logic* $\mathrm{FO(TC)}$ *captures nondeterministic logarithmic space* $\mathrm{NLOGSPACE}$.
- *On strings (word structures),* $\mathrm{SO(arity}\ k)(\mathrm{TC})$ *captures the complexity class* $\mathsf{NSPACE}(n^k)$.

In particular, on strings $\mathrm{MSO(TC)}$ captures nondeterministic linear space NLIN.

Existential positive $\mathrm{SO(2TC)}$

$\exists\mathrm{SO(2TC)}$ is the syntactic fragment of $\mathrm{SO(TC)}$ in which

1. the existential quantifiers and the TC-operators occur only positively.
2. TC-operators bound only second-order variables.

Rosen noted (99) that $\exists\mathrm{SO}$ collapses to existential first-order logic $\exists\mathrm{FO}$.

Theorem

The expressive powers of $\exists\mathrm{SO(2TC)}$ and $\exists\mathrm{FO}$ coincide.

Proof.

$$[\mathrm{TC}_{\vec{X},\vec{X}'}\exists x_1 \ldots \exists x_n \theta](\vec{Y},\vec{Y}') \text{ and } \mathfrak{A} \models [\mathrm{TC}^k_{\vec{X},\vec{X}'}\exists x_1 \ldots \exists x_n \theta](\vec{Y},\vec{Y}'),$$

where θ is quantifier free FO-formula, are equivalent for large enough k. (Note that k independent of the model in question and depends only on the formula.) □

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Corridor tiling problem

The *corridor tiling problem* is the following PSPACE-complete decision problem (Chlebus 86):
Input: An instance $P = (T, H, V, \vec{b}, \vec{t}, n)$ of the corridor tiling problem.
Output: Does there exist a corridor tiling for P? (Does there exists a tiling of width n having $\vec{b}$ as the first row and $\vec{t}$ as the last row?)

Complexity of model checking

Theorem

Combined complexity of model checking for monadic $2\mathrm{TC}[\forall\mathrm{FO}]$ *is* PSPACE*-complete.*

Proof.

Hardness follows from a reduction from corridor tiling. Input: $(T, H, V, \vec{b}, \vec{t})$. Let s be a successor relation on $\{0, 1, \ldots, n\}$ and $X_1, \ldots X_k, Y_1, \ldots, Y_k$ monadic second-order variables that correspond to tile types.

$$\varphi_H := \forall xy\big(s(x, y) \rightarrow \bigvee_{(i,j)\in H} Z'_i(x) \wedge Z'_j(y)\big), \quad \varphi_V := \forall x \bigvee_{(i,j)\in V} Z_i(x) \wedge Z'_j(x)$$

$$\varphi_T := \forall x \bigvee_{i\in T} \big(Z'_i(x) \wedge \bigwedge_{j\in T, i\neq j} \neg Z'_j(x)\big),$$

The formula $\mathrm{TC}_{\vec{Z},\vec{Z}'}[\varphi_T \wedge \varphi_H \wedge \varphi_V](\vec{X}, \vec{Y})$ describes proper tiling. $\square$

MSO(TC) and counting

- Assume a supply of *counter variables* μ and ν (with subscripts). Counters range over $\{0, \ldots, n\}$, where n is the cardinality the model.
- Assume a supply of k-ary *numeric predicates* $p(\mu_1, \ldots, \mu_k)$.
  - Intuitively relations over natural numbers such as the tables of multiplication and addition.
  - Technically similar to generalised quantifiers; a k-ary numeric predicate is a class $Q_p \subseteq \mathbb{N}^{k+1}$ of $k+1$-tuples of natural numbers.
  - When evaluating a k-ary numeric predicate $p(\mu_1, \ldots, \mu_k)$, the numeric predicate Q_p accesses also the cardinality of the structure in question.

MSO(TC) and counting

Definition

The syntax of CMSO(TC) extends the syntax of MSO(TC) as follows:

$$\varphi ::= (\mu = \#\{x : \varphi\}) \mid p(\mu_1, \ldots, \mu_k) \mid \exists \mu \varphi \mid [\mathrm{TC}_{\vec{X}, \vec{X}'} \varphi](\vec{Y}, \vec{Y}'),$$

where $\vec{X}$, $\vec{X}'$, $\vec{Y}$, and $\vec{Y}'$ may also include counter variables.

Semantics:

$\mathfrak{A} \models_s \mu = \#\{x : \varphi\}$ iff $s(\mu)$ equals the cardinality of $\{a \in A \mid \mathfrak{A} \models_{s(x \mapsto a)} \varphi\}$.

$\mathfrak{A} \models_s p(\mu_1, \ldots, \mu_k)$ iff $(|A|, s(\mu_1), \ldots, s(\mu_k)) \in Q_p$

$\mathfrak{A} \models_s \exists \mu \varphi$ iff there exists $i \in \{0, \ldots, n\}$ such that $\mathfrak{A} \models_{s(\mu \mapsto i)} \varphi$.

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Counting in NLOGSPACE

Definition

A k-ary numeric predicate Q_p is *decidable in* $\mathrm{NLOGSPACE}$ if the membership $(n_0, \ldots, n_k) \in Q_p$ can be decided by a nondeterministic Turing machine that uses logarithmic space when the numbers $n_0, \ldots, n_k$ are given in unary. Note that this is equivalent to linear space when $n_0, \ldots, n_k$ are given in binary.

We restrict to numeric predicates that are decidable in $\mathrm{NLOGSPACE}$.

Example

Let k be a natural number, $X, Y, Z, X_1, \ldots, X_n$ monadic second-order variables. The following numeric predicates are clearly $\mathrm{NLOGSPACE}$-definable:

- $\mathfrak{A} \models_s \mathrm{size}(X, k)$ iff $|s(X)| = k$,
- $\mathfrak{A} \models_s \times(X, Y, Z)$ iff $|s(X)| \times |s(Y)| = |s(Z)|$,
- $\mathfrak{A} \models_s +(X_1, \ldots, X_n, Y)$ iff $|s(X_1)| + \cdots + |s(X_n)| = |s(Y)|$.

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Counting in NLOGSPACE

Proposition (Immerman 87)

For every k-ary numeric predicate Q_p decidable in $\mathrm{NLOGSPACE}$ there exists a formula φ_p of $\mathrm{FO(TC)}$ over $\{s, x_1, \ldots, x_k\}$,

$$\mathfrak{A} \models_s p(\mu_1, \ldots, \mu_k) \text{ iff } \mathfrak{B} \models_t \varphi_p,$$

where $B = \{0, 1, \ldots, |A|\}$, $t(s)$ is the successor relation of B, and $t(x_i) = s(\mu_i)$, for $1 \leq i \leq k$.

$\mathrm{MSO(TC)}$ (without order) simulates $\mathrm{FO(TC)}$ with order

The idea is that natural numbers i are simulated by sets of cardinality i. Recall that $\mathrm{MSO(TC)}$ can express the Härtig quantifier!

The translation $^+ : \mathrm{FO(TC)} \to \mathrm{MSO(TC)}$ is defined as follows:

- For ψ of the form $x_i = x_j$, define $\psi^+ := \mathrm{H}xy\big(X_i(x), X_j(y)\big)$.
- For ψ of the form $s(x_i, x_j)$, define $\psi^+ := \exists z\Big(\neg X_i(z) \wedge \mathrm{H}xy\big(X_i(x) \vee x = z, X_j(y)\big)\Big)$.
- For ψ of the form $\exists x_i \varphi$, define $\psi^+ := \exists X_i \varphi^+$.
- For ψ of the form $[\mathrm{TC}_{\vec{x},\vec{x}'}\varphi](\vec{y}, \vec{y}')$, define $\psi^+ := [\mathrm{TC}_{\vec{X},\vec{X}'}\varphi^+](\vec{Y}, \vec{Y}')$.

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The translation ${}^{+} : \mathrm{FO(TC)} \rightarrow \mathrm{MSO(TC)}$ is defined as follows:

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$\mathrm{MSO(TC)}$ simulates $\mathrm{CMSO(TC)}$

In $\mathrm{MSO(TC)}$ counter variables are treated as set variables. Define a translation $* : \mathrm{CMSOTC} \to \mathrm{MSO(TC)}$.

- For an $\mathrm{NLOGSPACE}$ numeric predicate Q_p and ψ of the form $p(\mu_1, \ldots, \mu_k)$, define ψ^* as $\varphi_p^+(\mu_1/X_1, \ldots, \mu_k/X_k)$, where $^+$ is the translation defined above and φ_p the defining $\mathrm{FO(TC)}$ formula of Q_p.
- For ψ of the form $\mu = \#\{x \mid \varphi\}$, ψ^* is $\mathrm{H}xy(\varphi^*, \mu(y))$.
- For ψ of the form $\exists \mu_i \varphi$, define ψ^* as $\exists \mu_i \varphi^*$.

Order invariant MSO

A formula $\varphi \in \mathrm{MSO}$ over $\tau_{\leq}$ is *order-invariant*, if for every τ-structure $\mathfrak{A}$ and expansions $\mathfrak{A}'$ and $\mathfrak{A}^*$ of $\mathfrak{A}$ to the vocabulary $\tau_{\leq}$, in which $\leq^{\mathfrak{A}'}$ and $\leq^{\mathfrak{A}^*}$ are linear orders of A, we have that

$$\mathfrak{A}' \models \varphi \text{ if and only if } \mathfrak{A}^* \models \varphi.$$

A class $\mathcal{C}$ of τ-structures is *definable in order-invariant* MSO if and only if the class

$$\{(\mathfrak{A}, \leq) \mid \mathfrak{A} \in \mathcal{C} \text{ and } \leq \text{ is a linear order of } A\}$$

is definable by some order-invariant MSO-formula.

Order invariant MSO and MSO(TC)

Example

Consider the class
$$\mathcal{C} = \{\mathfrak{A} \mid |A| \text{ is a prime number}\}$$
of $\emptyset$-structures. The language of prime length words over some unary alphabet is not regular and thus it follows via Büchi's theorem that $\mathcal{C}$ is not definable in order-invariant MSO. However the following formula of MSO(TC) defines $\mathcal{C}$.

$$\exists X \forall Y \forall Z \big(\forall x (X(x)) \wedge (\mathrm{size}(Y,1) \vee \mathrm{size}(Z,1) \vee \neg \times (Y,Z,X))\big) \wedge \exists x \exists y \, \neg x = y.$$

Corollary

For any vocabulary τ, there exists a class $\mathcal{C}$ of τ-structures such that $\mathcal{C}$ is definable in MSO(TC) but it is not definable in order-invariant MSO.

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Theorem

Over finite unary vocabularies $\mathrm{MSO(TC)}$ *is strictly more expressive than order-invariant* MSO*.*

Proof.

The proof is based on Parikh's Theorem (66):
For every regular language L its Parikh image (letter count) $P(L)$ is a finite union of linear sets. $\square$

A subset S of $\mathbb{N}^k$ is a *linear set* if

$$S = \{\vec{v}_0 + \sum_{i=1}^{m} a_i \vec{v}_i \mid a_1, \ldots, a_m \in \mathbb{N}\}$$

for some *offset* $\vec{v}_0 \in \mathbb{N}^k$ and *generators* $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{N}^k$.

Order invariant MSO and MSO(TC)

Theorem

Over finite unary vocabularies $\mathrm{MSO(TC)}$ *is strictly more expressive than order-invariant* MSO.

Proof.

The proof is based on Parikh's Theorem (66):
For every regular language L its Parikh image (letter count) $P(L)$ is a finite union of linear sets. □

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Open question

- Does the exists a formula of least fixed point logic LFP that is not expressible in $\mathrm{MSO(TC)}$. On ordered structures, this would show that there are problems in P that are not in NLIN, which is open (it is only know that the two classes are different).
- Note that EVEN is definable in $\mathrm{MSO(TC)}$ but not in LFP (over empty vocabulary).
- What is the relationship of $\mathrm{MSO(TC)}$ and order-invariant MSO over vocabularies of higher arity?

Happy Birthday Lauri!