Finite model theory

1. Models

The notion of a model

Algebra and discrete mathematics study *mathematical structures*. For example the group $(\mathbb{Z}, +)$ and the ring $(\mathbb{Z}, +, \cdot)$ are structures.

The notion of a **model** formalizes the idea of a structure.

To define the notion of a model formally, let us first recap the notions of a **relation** and **function**.

Relations

Let $\mathbb{Z}_+$ denote the positive integers, i.e., $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. If X is a set, recall that X^n denotes the n-fold Cartesian product of X with itself, i.e.,

$$X^n = \{\ (x_1, \dots, x_n)\ \mid\ x_1, \dots, x_n \in X\ \}.$$

Definition 1.1

*An n-ary **relation** over a set X is any set $R \subseteq X^n$.*

In other words, an n-ary relation over X is any set R of tuples $(x_1, \dots, x_n)$ of elements of X. The number n is also called the **arity** of R.

Relations

Example 1.2

The set

$$P = \{ \ x \in \mathbb{R} \mid x > 0 \ \}$$

is the unary (i.e., 1-ary) relation over $\mathbb{R}$ that contains precisely the positive real numbers. The set

$$Q = \{ \ x \in \mathbb{R} \mid x \in \mathbb{Q} \ \}$$

is the unary relation over $\mathbb{R}$ containing precisely the rationals.

Relations

Example 1.3

The set

$$S = \{\ (x, y) \in \mathbb{R}^2 \mid x \leq y\ \}$$

is the binary (i.e., 2-ary) relation over $\mathbb{R}$ containing the pairs (x, y) such that $x \leq y$. Clearly S is the 'smaller or equal to' relation over $\mathbb{R}$. Note that in the literature, the symbol $\leq$ can sometimes be identified with the corresponding relation S, so we could write, say, that $(2, 3) \in \leq$ and $(10, 1) \notin \leq$.

Relations

Example 1.4

Let $E = \mathbb{R}^2$ denote the Euclidean plane. The set

$$P = \{ \ (x, y, z) \in E^3 \mid x + y = z \ \}$$

is the ternary (i.e., 3-ary) relation over $E = \mathbb{R}^2$ containing precisely those triples (x, y, z) of vectors x, y, z in E such that $x + y = z$. Thus P corresponds to vector summation in the Euclidean plane.

Relations

Binary relations are perhaps the most common type of relation in the literature.

Let R be a binary relation over X, i.e., $R \subseteq X^2 = X \times X$.

1. R is **reflexive** if $(x, x) \in R$ for all $x \in X$.
   R is **irrefleksive** if $(x, x) \notin R$ holds for all $x \in X$.
2. R on **symmetric** if we have
   $$(x, y) \in R \Rightarrow (y, x) \in R$$
   for all $x, y \in R$.

   R on **antisymmetric** if
   $$\forall x, y \in X\Big(\ ((x, y) \in R \text{ and } (y, x) \in R\,) \ \Rightarrow \ x = y\Big).$$
3. R on **transitive** if
   $$\forall x, y, z \in X\Big(\ ((x, y) \in R \text{ and } (y, z) \in R\,) \ \Rightarrow \ (x, z) \in R\,\Big).$$

Example 1.5

The figure shows a binary relation $R = \{(a, b), (b, c), (c, b)\}$ over the set $\{a, b, c\}$. This relation is not symmetric because we have $(a, b) \in R$ but $(b, a) \notin R$. However, R is not antisymmetric either, because we have $(b, c), (c, b) \in R$ but $b \neq c$.

Relations

Definition 1.6
Let $R, S \subseteq X \times X$ be binary relations. The **inverse relation** of R is the relation

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

The **composition** of R ja S is the relation

$$R \circ S = \{(x, z) \in X \times X \mid \text{there exists } y \in X \text{ such that } (x, y) \in R \text{ ja } (y, z) \in S\}.$$

It is easy to show that $R \circ (S \circ T) = (R \circ S) \circ T$.

Relations

Lemma 1.7

Consider $\bigcup_{n \in \mathbb{Z}_+} R^n$ *where* $R^1 = R$ *and* $R^{n+1} = R \circ R^n$. *The relation* $\bigcup_{n \in \mathbb{Z}_+} R^n$ *is transitive.*

Proof. Let us first prove that $R^m \circ R^n = R^{m+n}$ for positive integers m, n. We prove this by induction on m. When $m = 1$, we have $R^1 \circ R^n = R \circ R^n$ and this equals R^{n+1} by definition, so the base case is clear. Assume then that $R^k \circ R^n = R^{k+n}$. Now $R^{k+1} \circ R^n = (R \circ R^k) \circ R^n = R \circ (R^k \circ R^n) = R \circ (R^{k+n}) = R^{k+n+1}$, so we are done.

Let us then prove $\bigcup_{n \in \mathbb{Z}_+} R^n$ transitive. Suppose $(x, y), (y, z) \in \bigcup_{n \in \mathbb{Z}_+} R^n$. Thus $(x, y) \in R^m$ and $(y, z) \in R^n$ for some n and m. Therefore we have $(x, z) \in R^m \circ R^n = R^{m+n}$. As $R^{m+n} \subseteq \bigcup_{n \in \mathbb{Z}_+} R^n$, we have $(x, z) \in \bigcup_{n \in \mathbb{Z}_+} R^n$, so we are done.$\square$

Relations

Definition 1.8

Consider a binary relation $R \subseteq X \times X$. The **transitive closure** $\mathrm{TR}(R)$ is the smallest transitive set T such that $R \subseteq T$. More formally, the transitive closure $\mathrm{TR}(R)$ is the transitive set T such that

1. $R \subseteq T$ and
2. $T \subseteq V$ for all transitive sets such that $R \subseteq V$.

Example 1.9

Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$. Consider the binary relation

$$S = \{(s,t) \in \mathbb{N}^2 \mid t = s + 1\}.$$

The relation S is called the *successor relation* over $\mathbb{N}$. The transitive closure $\mathrm{TC}(S)$ of S is the *strict linear order* over $\mathbb{N}$, i.e., the relation

$$< \; = \; \{(s,t) \in \mathbb{N}^2 \mid s < t\}.$$

We sometimes denote this relation by $<^{\mathbb{N}}$.

Relations

Theorem 1.10

Let $R \subseteq X \times X$ be a binary relation. Let F be the set of transitive sets $T \subseteq X \times X$ such that $R \subseteq T$. Then

$$\mathrm{TC}(R) = \bigcap F.$$

On the other hand,

$$\mathrm{TC}(R) = \bigcup_{n \in \mathbb{Z}_+} R^n$$

where $R^1 = R$ and $R^{n+1} = R \circ R^n$.

Proof. We first prove that $\mathrm{TC}(R) = \bigcap F$ by proving that $\bigcap F$ is the smallest set that is transitive and contains R as a subset. We begin by noting that $F \neq \emptyset$ because $X \times X \in F$. Thus $\bigcap F$ is a defined set.

We then note that $\bigcap F$ is transitive:

$$\begin{aligned}(x,y),(y,z)\in\bigcap F &\Rightarrow (x,y),(y,z)\in T \text{ for all } T\in F\\ &\Rightarrow (x,z)\in T \text{ for all } T\in F\\ &\Rightarrow (x,z)\in\bigcap F.\end{aligned}$$

As $\bigcap F$ is the intersection of all transitive sets containing R as a subset, it is now clear that $\bigcap F$ is the smallest transitive set containing R. Thus we have shown that $\mathrm{TC}(R)=\bigcap F$.

We then prove that $\mathrm{TC}(R)=\bigcup_{n\in\mathbb{Z}_+} R^n$. Lemma 10 shows that $\bigcup_{n\in\mathbb{Z}_+} R^n$ is transitive. Thus it now suffices to show than for all transitive sets S containing R, the set S also contains $\bigcup_{n\in\mathbb{Z}_+} R^n$.

Thus let S be an arbitrary transitive set such that $R\subseteq S$. We show by induction that $R^n\subseteq S$ for all n. By definition of S, we have $R^1=R\subseteq S$. Assume then that $R^n\subseteq S$. Let $(x,z)\in R^{n+1}=R\circ R^n$. Thus $(x,y)\in R$ and $(y,z)\in R^n$ for some y. As $R,R^n\subseteq S$, we have $(x,y),(y,z)\in S$. As S is transitive, we have $(x,z)\in S$. $\square$

Example 1.11

Let P denote the 'parent relation' connecting a person x to a person y iff y is a parent of x. Then $\mathrm{TC}(P)$ is the 'ancestor relation' connecting x to y iff y is an ancestor of x.

Relations

Let us then consider some important types of relations.

Definition 1.12

Let R be a binary relation over X.

1. R is a **preorder** if it is reflexive and transitive.
2. R is a **partial order** if it is reflexive, transitive and antisymmetric.
3. R is a **weak linear order** if it is reflexive, transitive, antisymmetric and satisfies the following *comparativity condition*:

   for all $x, y \in X$, we have $(x, y) \in R$ or $(y, x) \in R$.

4. R is a **strict linear order** if it is irreflexive, transitive and satisfies the following *trichotomy condition*:

   for all $x, y \in X$, exactly one of the following holds:
   - $(x, y) \in R$,
   - $(y, x) \in R$,
   - $x = y$.

Relations

The 'less or equal than' relation $\leq$ over the reals/rationals/integers is an example of a weak linear order, and the corresponding relation $<$ is a strict linear order

In the literature, a 'linear order' can mean a weak linear order or a strict linear order. The context will dictate which one is meant.

Definition 1.13

Let R be a binary relation over X. Then R is an **equivalence relation** if it is reflexive, transitive and symmetric. The relation R is an **identity relation** (also known as an **equality relation**) if $R = \{ (x, x) \mid x \in X \}$.

Example 1.14

Consider the binary relation B of 'having been born the same year', i.e., for any people x and y, we have $(x, y) \in B$ iff x and y were born the same year. This relation B is an equivalence relation.

Functions

Let us recap the basic definitions relating to **functions**, also known as **maps**.

Let A and B be sets and $n \in \mathbb{Z}_+$. A function $f : A^n \to B$ associates every tuple $(a_1, \ldots, a_n) \in A^n$ with precisely one element $f(a_1, \ldots, a_n) \in B$ of B. In particular, a unary (i.e., 1-ary) function $f : A \to B$ associates every element of $a \in A$ with precisely one element $f(a) \in B$ of B.

The set A^n is called the **domain** of $f : A^n \to B$ while B is the **codomain** of the function. The **range** of $f : A^n \to B$ is the set

$$\{\, f(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A \,\}.$$

Functions

An n-ary **partial function** f from A^n to B is a function $f : V \to B$ for some set $V \subseteq A^n$. In particular, a unary (i.e., 1-ary) partial function from A to B is a function $f : V \to B$ for some $V \subseteq A$.

Functions

Let $n \in \mathbb{Z}_+$. An n-ary function *over* a set X is a function $f : X^n \to X$ with domain X^n and codomain X.

An n-ary function over X is identical to the $(n+1)$-ary relation

$$\begin{aligned} & \{ \ ((x_1, \ldots, x_n), f(x_1, \ldots, x_n)) \mid x_1, \ldots, x_n \in X \ \} \\ = \ & \{ \ (x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \mid x_1, \ldots, x_n \in X \ \} \end{aligned}$$

over X. In other words, every n-ary function over X is an $(n+1)$-ary relation over X.

Functions

An n-ary partial function *over* X is a function $f : V \to X$ where $V \subseteq X^n$. In particular, a unary partial function over X is a function $f : V \to X$ for some $V \subseteq X$.

Functions

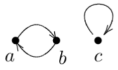
Definition 1.15

Let $f : A \to B$ be a unary function. The function f is an **injection** if for all different x and y in A, we have $f(x) \neq f(y)$. In other words, different elements never map to the same element. The function f is a **surjection** if for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$. In other words, every element b of the codomain B has some element mapping to it. The function f is a **bijection** if it is an injection and a surjection.

Functions

Let $f : A \to A$ be a unary function over A. An element $a \in A$ is called a **fixed point** (or a **fixpoint**) of f if $f(a) = a$.

Example 1.16



The figure shows a binary relation $R = \{(a, b), (b, a), (c, c)\}$ over the set $\{a, b, c\}$. The relation R is also a unary function f with a fixed point c, i.e., $f(c) = c$. The pair (c, c) is called a **reflexive loop** of R.

Models

We then begin discussing the notion of a model. We first provide an informal discussion of the notion. Consider the fields

$$(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1)$$

and

$$(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, 0, 1)$$

where

1. $+^{\mathbb{R}}$ is the binary function $+^{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ denoting summation of reals,
2. $\cdot^{\mathbb{R}}$ the binary function $\cdot^{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ denoting multiplication of reals,
3. $+^{\mathbb{Q}}$ the binary function $+^{\mathbb{Q}} : \mathbb{Q}^2 \to \mathbb{Q}$ denoting summation of rationals, and
4. $\cdot^{\mathbb{Q}}$ the binary function $\cdot^{\mathbb{Q}} : \mathbb{Q}^2 \to \mathbb{Q}$ denoting multiplication of rationals.

The fields $(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1)$ and $(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, 0, 1)$ are examples of models.

Models

Also the *ordered fields*

$$(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1, \leq^{R})$$

and

$$(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, 0, 1, \leq^{Q})$$

are models. Here

$$\leq^{\mathbb{R}} = \{ (x, y) \in \mathbb{R}^2 \mid x \leq y \} \text{ and}$$

$$\leq^{\mathbb{Q}} = \{ (x, y) \in \mathbb{Q}^2 \mid x \leq y \}$$

are the binary linear order relations over $\mathbb{R}$ and $\mathbb{Q}$, respectively.

Models

Intuitively, models are mathematical entities that encode enough information about some mathematical realm of interest. For example, the model

$$(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1)$$

enables us to do arithmetic with real numbers. Similarly,

$$(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{R}}, 0, 1)$$

enables arithmetic with rationals.

However, models have also further uses outside classical mathematics. For example, *databases* are typically identified with relational models, i.e., models consisting of relations.

Models

Having defined the notion of a model informally, we now begin working towards a formal definition. To that end, we next discuss the notion of a **symbol**. There are three types of symbols: **function symbols, relation symbols** and **constant symbols.**

Function symbols are typically symbols such as f, g, h, et cetera. Each function symbol is associated with an arity $n \in \mathbb{Z}_+$.

Relation symbols are typically symbols such as R, S, T, P et cetera. Each relation symbol is associated with an arity $n \in \mathbb{Z}_+$.

Constant symbols are typically symbols such as c, d, e et cetera. We do not associate an arity with constant symbols, although they are often considered to be nullary (i.e., 0-ary) function symbols in the literature.

Models

A **vocabulary** is a set of symbols. A vocabulary can also be called a **signature**. An **algebraic vocabulary** consists of function and constant symbols only. A **relational** vocabulary consists of relation symbols and constant symbols only. (We note that in the literature, a 'relational vocabulary' is sometimes not allowed to contain constant symbols.) A **purely relational** vocabulary refers to relational vocabularies without constant symbols.

Vocabularies are typically denoted by τ or σ (where σ relates to the term 'signature').

Models

Definition 1.17

Let τ be a vocabulary. A τ-model $\mathfrak{M}$ is a pair (M, T) where

1. M is a nonempty set, called the **domain** of the model,
2. T is function with domain τ that maps
   - each n-ary relation symbol $R \in \tau$ to an n-ary relation $R^{\mathfrak{M}} \subseteq M^n$ over M,
   - each n-ary function symbol $f \in \tau$ to an n-ary function $f^{\mathfrak{M}} : M^n \to M$ over M,
   - each constant symbol $c \in \tau$ to some element $c^{\mathfrak{M}} \in M$.

Models

Models are typically denoted by, e.g., the symbols $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}$ et cetera, and we adopt the convention that the corresponding domain sets are then A, B, M et cetera. The domain M of a model $\mathfrak{M}$ can also be denoted by $dom(\mathfrak{M})$.

Note, it may be inconvenient to use the fraktur letters (e.g., $\mathfrak{M}$) when writing models by hand (on paper or on the blackboard). In general it does not matter how models are written by hand, as long as it is clear from the context what means what. For example calligraphic symbols (e.g., $\mathcal{M}$) may be more convenient for writing by hand.

Models

Example 1.18

Recall the model $(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1, \leq^{\mathbb{R}})$. Strictly speaking, we should write this model as a pair $\mathfrak{R} = (\mathbb{R}, T)$ where the map T has the vocabulary $\{+, \cdot, 0, 1, \leq\}$ as its domain such that

1. $T(+) = +^{\mathfrak{R}}$ is the binary summation function over the $\mathbb{R}$. Note here that while $+$ is a function symbol, we let $+^{\mathfrak{R}}$ denote the corresponding function.
2. $T(\cdot) = \cdot^{\mathfrak{R}}$ is the binary multiplication function over $\mathbb{R}$. here $\cdot$ is a function symbol and $\cdot^{\mathfrak{R}}$ denotes the corresponding function.
3. $T(0) = 0^{\mathfrak{R}} \in \mathbb{R}$ is the constant zero. Here 0 is a constant symbol while $0^{\mathfrak{R}}$ is the corresponding constant in $\mathbb{R}$.
4. $T(1) = 1^{\mathfrak{R}} \in \mathbb{R}$ is the constant one. Here 1 is a constant symbol while $1^{\mathfrak{R}}$ is the corresponding constant in $\mathbb{R}$.
5. $T(\leq) = \leq^{\mathfrak{R}} = \{ (x, y) \in \mathbb{R}^2 \mid x \leq y \}$, so the first symbol $\leq$ is a relation symbol and $\leq^{\mathfrak{R}}$ denotes the corresponding relation.

Models

We rarely need to be as formal as in the above example. Typically it would suffice to talk about the model $(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1, \leq^{\mathbb{R}})$ instead of the pair $(\mathfrak{R}, T)$. We could even refer to this structure as the model $\mathbb{R}$ as long as it is clear from the context what structure we actually mean.

Similarly, it is not always necessary to differentiate between a relation symbol R and the related relation $R^{\mathfrak{M}}$. As long as it is fully clear what is meant from the context, we may use the symbol R to denote the relation $R^{\mathfrak{M}}$. The same goes for function symbols and constant symbols.

Models

Let $\tau = \{R\}$ be a vocabulary with one binary relation symbol R. The figure shows a model $(\{a, b, c\}, T)$ where $T(R) = \{(a, b), (b, c), (c, b)\}$. It is also ok to define this as, say, a model $\mathfrak{M} = (M, R^{\mathfrak{M}})$ where $M = \{a, b, c\}$ and $R^{\mathfrak{M}} = \{(a, b), (b, c), (c, b)\}$. It is even ok to define it as a model $\mathfrak{M} = (M, R)$ where M is as above and $R = \{(a, b), (b, c), (c, b)\}$. So here we are indeed not differentiating between relations and relation symbols.

We note, however, that it is sometimes convenient to have the fully formal definition of a model at hand.

Models

Example 1.19

Consider once more the models $(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1, \leq^{\mathbb{R}})$ and $(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, 0, 1, \leq^{\mathbb{Q}})$. These models have the same vocabulary $\{+, \cdot, 0, 1, \leq\}$ with two function symbols, two constant symbols and one relation symbol. The domains $\mathbb{R}$ and $\mathbb{Q}$ are different.

The models $(\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0, 1)$ and $(\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, 0, 1)$ are similarly related, this time the vocabulary being $\{+, \cdot, 0, 1\}$.

Each of these four models of the example here have an infinite domain.

Models

Definition 1.20

A finite model is a model whose domain is finite, and furthermore, also the vocabulary of the model is finite.

Finite model theory studies finite models with a relational vocabulary, i.e., finite relational models. Typically the models studied are actually purely relational, so constant symbols are out of the picture. Function symbols are not considered. However, this is not a significant limitation, since (as we have observed above) n-ary functions are a special case of $(n+1)$-ary relations.

Models

The **cardinality** of a model $\mathfrak{M}$, denoted $card(\mathfrak{M})$, is the number of elements in the domain M of $\mathfrak{M}$. That is, $card(\mathfrak{M}) = |M|$.

The elements $x \in M$ in the domain of a model $\mathfrak{M}$ have various names. They can be called, e.g., **elements**, **points** or **nodes**.

Definition 1.21

A **directed graph** is a model (V, E) where V is a finite set of *vertices* (singular: vertex) and $E \subseteq V \times V$ is a binary relation that does not contain reflexive loops.

Thus, in a directed graph (V, E), there are no vertices $v \in V$ such that $(v, v) \in E$.

Definition 1.22

A **graph** is a directed graph (V, E) where E is symmetric, i.e., $(u, v) \in E \Rightarrow (v, u) \in E$.

Morphisms

One of the most important notions in mathematics is that of *similarity*. Pairs of structures have different levels of similarity, and this phenomenon is conveniently studied in terms of different kinds of morphisms.

Morphisms

The first kind of morphism that we study is **homomorphism**.

Definition 1.23

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-models, i.e., models over the vocabulary τ. Let $h : A \to B$ be a function mapping from the domain A of $\mathfrak{A}$ to the domain B of $\mathfrak{B}$. The function h is a **homomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if h satisfies the following **homomorphism conditions**:

1. $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constant symbols $c \in \tau$.
2. $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$ for all n-ary function symbols $f \in \tau$ and all $a_1, \ldots, a_n \in A$.
3. $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Rightarrow (h(a_1), \ldots, h(a_n)) \in R^{\mathfrak{B}}$ for all $a_1, \ldots, a_n \in A$ and all n-ary relation symbols $R \in \tau$.

Morphisms

Definition 1.24

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-models. Let $h : A \to B$ be a function mapping from the domain A of $\mathfrak{A}$ to the domain B of $\mathfrak{B}$. The function h is a **strong homomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if h satisfies the following **strong homomorphism conditions**:

1. $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constant symbols $c \in \tau$.
2. $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$ for all n-ary function symbols $f \in \tau$ and all $a_1, \ldots, a_n \in A$.
3. $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (h(a_1), \ldots, h(a_n)) \in R^{\mathfrak{B}}$ for all $a_1, \ldots, a_n \in A$ and all n-ary relation symbols $R \in \tau$.

The difference to homomorphism is that now the condition 3 has a bi-implication $\Leftrightarrow$ instead of implication $\Rightarrow$.

Morphisms

A strong homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$ is

1. an **embedding** if h is an injection,
2. an **isomorphism** if h is a bijection.

Two models $\mathfrak{A}$ and $\mathfrak{B}$ are said to be **isomorphic**, denoted $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$. Intuitively this means that $\mathfrak{A}$ and $\mathfrak{B}$ are essentially the same; we obtain $\mathfrak{B}$ from $\mathfrak{A}$ by *replacing* the domain elements a of $\mathfrak{A}$ with the elements $h(a)$ of $\mathfrak{B}$ but keeping the structure otherwise the same. An isomorphism h from $\mathfrak{A}$ to $\mathfrak{A}$ itself is called an **automorphism**. A homomorphism h from $\mathfrak{A}$ to $\mathfrak{A}$ itself is called an **endomorphism**.

If there is an embedding h from $\mathfrak{A}$ to $\mathfrak{B}$, we say that $\mathfrak{A}$ **embeds into** $\mathfrak{B}$. Intuitively this means there exists a copy of the structure $\mathfrak{A}$ inside $\mathfrak{B}$.

Morphisms

Definition 1.25

Let $\mathfrak{A}$ be τ-model, where τ is a relational vocabulary. Then the **restriction** of $\mathfrak{A}$ to $B \subseteq A$ is the model $\mathfrak{B} = \mathfrak{A} \upharpoonright B$ (with domain B) defined such that

$$R^{\mathfrak{B}} = \{ (a_1, \ldots, a_n) \in B^n \mid (a_1, \ldots, a_n) \in R^{\mathfrak{A}} \}$$

for all relation symbols $R \in \tau$. It is also required that $c^{\mathfrak{B}} = c^{\mathfrak{A}}$ for all constant symbols $c \in \tau$. Note that this requires that all the constants $c^{\mathfrak{A}}$ of $\mathfrak{A}$ belong to the set B.

A τ-model $\mathfrak{C}$ is a **substructure** of $\mathfrak{A}$ if $\mathfrak{C}$ is some restriction $\mathfrak{A} \upharpoonright C$ of $\mathfrak{A}$ to some set $C \subseteq A$. It is easy to see that then $\mathfrak{C}$ embeds into $\mathfrak{A}$.

Definition 1.26

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-models for a relational vocabulary τ. A **partial isomorphism** p from $\mathfrak{A}$ to $\mathfrak{B}$ is an isomorphism $p : U \to V$ from $\mathfrak{A} \upharpoonright U$ to $\mathfrak{B} \upharpoonright V$ for some sets $U \subseteq A$ and $V \subseteq B$.

Thus the partial isomorphism p maps some substructure $\mathfrak{S}$ of $\mathfrak{A}$ onto a substructure $\mathfrak{T}$ of $\mathfrak{B}$ such that $\mathfrak{S} \cong \mathfrak{T}$.

Definition 1.27

Let $\mathfrak{M}$ and $\mathfrak{N}$ be τ-models for a purely relational vocabulary τ. The **disjoint union** of $\mathfrak{M}$ and $\mathfrak{N}$ is the τ-model $\mathfrak{M} \uplus \mathfrak{N}$ with domain $(M \times \{0\}) \cup (N \times \{1\})$ and with each n-ary $R \in \tau$ specified such that

$$R^{\mathfrak{M} \uplus \mathfrak{N}}$$

$$= \{ ((a_1, x), \ldots, (a_n, x)) \mid (a_1, \ldots, a_n) \in (R^{\mathfrak{M}})^n \cup (R^{\mathfrak{N}})^n \text{ and } x \in \{0, 1\} \}.$$

Intuitively, the disjoint union is the model obtained by taking a copy of $\mathfrak{M}$ and and copy of $\mathfrak{N}$, and then putting these two models into the same model without letting the two domains overlap.

Definition 1.28

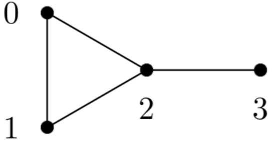
Let R be a binary relation and consider a model $\mathfrak{M} = (M, R^{\mathfrak{M}})$. Let $A \subseteq M$ be a nonempty set. The **generated submodel** of $\mathfrak{M}$ generated by A is the model $\mathfrak{M} \upharpoonright N$ where the domain N of $\mathfrak{M} \upharpoonright N$ is the smallest set defined as follows.

1. $A \subseteq N$,
2. If $a \in N$ and $(a, b) \in R^{\mathfrak{M}}$, then $b \in N$.

Intuitively, this generated submodel of $\mathfrak{M}$ is the submodel defined by restricting to all points that are accessible (possibly with several steps) from the points of A via $R^{\mathfrak{M}}$.

Notes on some further types of models

Recall from Definition 1.22 that a graph is directed graph (cf. Definition 1.21) (V, E) where the binary relation E is symmetric. Due to this symmetricity, we ofted draw graphs somewhat differently from standard models where the vocabulary consists of one binary relation. Consider, for example, the following figure.

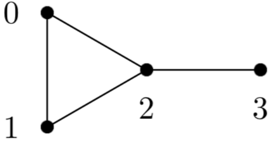


$\mathbb{G}$

The figure shows a graph $\mathbb{G}$ with domain $V = \{0, 1, 2, 3\}$ and binary relation $E = \{ (0,1), (1,0), (0,2), (2,0), (1,2), (2,1), (2,3), (3,2) \}$. Typically when drawing models, the pair of tuples $(0,1), (1,0)$ would correspond to two arrows, one from 0 to 1 and the other one back from 1 to 0. It is always ok to draw models that way, but when drawing graphs, it may be handy to simply draw lines connecting points rather than double arrows. This simplifies the drawings.

Notes on some further types of models

In standard graph theory, the graph



$\mathbb{G}$

would typically be specified as the pair

$$(\{0,1,2,3\},\{\{0,1\},\{0,2\},\{1,2\},\{2,3\}\}).$$

Then our pair of tuples $(0,1),(1,0)$ would be replaced by the set $\{0,1\}$, and so on. Each pair $\{0,1\}$ would be called an 'edge'.

In this course, we will stick to the definition of $\mathbb{G}$ from the previous slide, but it is good to know that conventions vary in the literature. It is also worth noting that in general, tuples $(a_1,\ldots,a_n)$ of an n-ary relation R can often be called edges, or n-ary edges. In particular, tuples (u,v) of a directed graph can be called edges. Typically no confusion with the definitions of standard graph theory arises, as it is clear from the context what is meant.

Notes on some further types of models

Let I be a set of indices and R a binary relation symbol. Define the vocabulary $\tau = \{P_i \mid i \in I\} \cup \{R\}$ where each P_i is a unary relation symbol. A **Kripke structure** over τ is a model $(W, R, (P_i)_{i \in I})$. Elements $w \in W$ of the domain W are often called 'worlds' when discussing Kripke structures. This relates to the original uses of Kripke structures in work on modal logic. Note that W need not be finite.

A **labelled transition system** is a Kripke structure. Labelled state transition systems are used in computer science to describe systems evolving in time. The points in the domain of a labelled transition system are called 'states'. For a state u of a transition system $\mathfrak{S}$, if $u \in P_i^{\mathfrak{S}}$, then the state u is interpreted to 'satisfy the property' P_i. For states u and v, if $(u, v) \in R^{\mathfrak{S}}$, then v is considered to be a state that is 'accessible' from u in one time step.

Note that the index set I above may be empty, and in that case we have a Kripke structure (or a state transition system) with the vocabulary $\{R\}$. Directed graphs are such Kripke structures. (The relation symbol for directed graphs is typically E instead of R, but that is not a significant issue.)

Notes on some further types of models

Let (V, R) be a model, $R \subseteq V \times V$. A **directed walk** is a sequence $(v_1, v_2, \dots)$ of points $v_1, v_2, ... \in V$ such that for each v_i, v_{i+1}, we have $(v_i, v_{i+1}) \in R$. If the sequence is of type $(v_1, \dots, v_n)$, then we say that the walk is finite. Otherwise the sequence is said to be an ω-**sequence**, and then we can identify the sequence with a function $f : \mathbb{Z}_+ \to V$ such that $f(i) = v_i$.

A directed walk is a **directed path** if all the points in the sequence are different. A **directed cycle** is a finite sequence $(v_1, \dots, v_n, v_1)$ where $(v_1, \dots, v_n)$ is a directed path and $(v_n, v_1) \in R$. (Here $n \geq 1$, so the shortest directed cycle is of type (v, v), i.e., a reflexive loop.)

Notes on some further types of models

In a model (V, R) with $R \subseteq V \times V$, an **undirected walk** is a nonempty sequence $((v_{1,1}, v_{1,2}), (v_{2,1}, v_{2,2}), \ldots, (v_{i,1}, v_{i,2}), \ldots)$ of tuples in R such that

1. for each $(v_{i,1}, v_{i,2})$, we have $(v_{i,1}, v_{i,2}) \in R \cup R^{-1}$ and
2. for all $(v_{i,1}, v_{i,2})$ and $(v_{i+1,1}, v_{i+1,2})$, we have $v_{i,2} = v_{i+1,1}$.

If the sequence is of type

$$((v_{1,1}, v_{1,2}), (v_{2,1}, v_{2,2}), \ldots, (v_{i,1}, v_{i,2}), \ldots, (v_{n,1}, v_{n,2})),$$

then the walk is finite. Otherwise it is an ω-**sequence** and we can identify the sequence with a function $f : \mathbb{Z}_+ \to R$ in the natural way.

A directed walk is a **directed path** if all the tuples $(v_{i,1}, v_{i,2})$ in the sequence have differing sets $\{v_{i,1}, v_{i,2}\}$ of nodes, no tuple is of type (v, v), and a point u can occur in two different tuples $(v_{i,1}, v_{i,2})$ and $(v_{j,1}, v_{j,2})$ only if $j = i + 1$ or $i = j + 1$.

Notes on some further types of models

A model (V, R) with $R \subseteq V \times V$ is **connected** if for all different points $v, u \in V$, there is an an undirected path

$$((v_{1,1}, v_{1,2}), \ldots, (v_{n,1}, v_{n,2}))$$

such that $u = v_{1,1}$ and $v = v_{n,2}$.

The model (V, R) is **strongly connected** if for all different $u, v \in V$, there is a directed path $(u, \ldots, v)$.

Notes on some further types of models

A directed graph $G = (V, E)$ is said to be **acyclic** if there does not exist an undirected path $((u, v), \ldots, (u', v'))$ such that $(v', u) \in R \cup R^{-1}$. The structure G is also called a **tree** if it is acyclic.

Notes on some further types of models

Consider a τ-model $\mathfrak{M} = (M, T)$ where T is the function giving interpretations to the symbols in the vocabulary τ. Let $\sigma \subseteq \tau$. Then the σ-model $\mathfrak{N} = (M, T \restriction \sigma)$ is called the **σ-reduct** of $\mathfrak{M}$. Intuitively, the reduct $\mathfrak{N}$ is otherwise the same as the orginal model $\mathfrak{M}$, but gives interpretations to only some of the symbols of $\mathfrak{M}$.

Conversely $\mathfrak{M}$ is called the **expansion** of $\mathfrak{N}$ to the vocabulary τ. (Obviously there can be several expansions of a model to the same larger vocabulary, but reducts are unique once the vocabulary of the reduct is fixed.)

We note that if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, then $\mathfrak{B}$ is called an **extension** of $\mathfrak{A}$ rather than an expansion. Extensions preserve the vocabulary while expansions generally do not.

Notes on some further types of models

Let $\sigma = \tau \cup \{\leq\}$ be a vocabulary. A σ-model interpreting $\leq$ as a weak linear order is called an **ordered model**.

Theorem 1.29

Let $\mathfrak{M}$ and $\mathfrak{N}$ be finite, ordered σ-models. Then there exists at most one isomorphism $f : M \to N$ between the models.

Proof. Suppose there exists some isomorphism $f : M \to N$ from $\mathfrak{M}$ to $\mathfrak{N}$. Let $M = \{m_1, \ldots, m_k\}$ and $N = \{n_1, \ldots, n_\ell\}$ such that

- $m_i \leq^{\mathfrak{M}} m_j \quad \Leftrightarrow \quad i \leq j$,
- $n_i \leq^{\mathfrak{N}} n_j \quad \Leftrightarrow \quad i \leq j$.

Since f is a bijection, we must have $\ell = k$. Therefore $N = \{n_1, \ldots, n_k\}$. We will show that $f(m_i) = n_i$ for all i. This implies that f is indeed the only possible isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$.

Notes on some further types of models

We will show $f(m_i) = n_i$ by induction on i. First, if $f(m_1) = n_p$ for $p \neq 1$, then there is some $q > 1$ such that $f(m_q) = n_1$. Therefore $m_1 \leq^{\mathfrak{M}} m_q$ but $n_1 = f(m_q) \leq^{\mathfrak{N}} f(m_1) = n_p$ (with $p \neq 1$). Thereby $m_1 \leq^{\mathfrak{M}} m_q$ and $f(m_1) \not\leq^{\mathfrak{N}} f(m_q)$, violating the isomorphism condition. Therefore we have $f(m_1) = n_1$, as desired.

Suppose then that $f(m_k) = n_k$ up to some index k. Now, as f is a bijection, we have $f(m_{k+1}) = n_\ell$ for some $\ell \geq k+1$. Suppose, for contradiction, that $\ell > k+1$. Then $f(m_p) = n_{k+1}$ for some $p > k+1$. Therefore $m_{k+1} \leq^{\mathfrak{M}} m_p$ and $f(m_{k+1}) = n_\ell \not\leq^{\mathfrak{N}} n_{k+1} = f(m_p)$, directly violating the isomorphism condition.$\square$

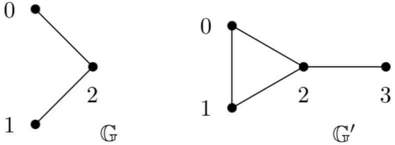
Notes on some further types of models

Recall that an isomorphism from a model $\mathfrak{M}$ to $\mathfrak{M}$ itself is called an automorphism. Theorem 1.29 shows that the only automorphism of a finite ordered model $\mathfrak{M}$ is the **trivial automorphism** $f : M \to M$ with $f(m) = m$ for all $m \in M$.

Definition 1.30

A model $\mathfrak{M}$ is called **rigid** if the only automorphism of the model is the trivial automorphism.

Notes on some further types of models

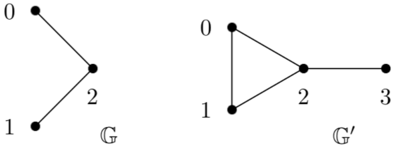


Example 1.31

Consider the graphs in the picture. It is easy to see that the graphs are not isomorphic. The simplest way to justify this is by noting that there trivially cannot be a bijection between the domains of different sizes (cardinalities).

It is easy to define homomorphisms from $\mathbb{G}$ to $\mathbb{G}'$, for example the maps $\{(0,0),(1,1),(2,2)\}$, $\{(0,1),(1,0),(2,2)\}$, $\{(0,0),(2,2),(1,3)\}$, and others.

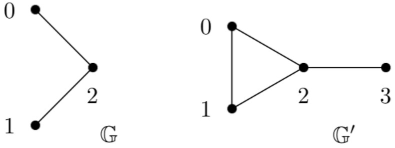
Notes on some further types of models



Note, however, that that these three homomorphisms specified above are actually embeddings of $\mathbb{G}$ into $\mathbb{G}'$. A somewhat more nontrivial homomorphism is the map $\{(0,0),(2,2),(1,0)\}$. This map is not an injection.

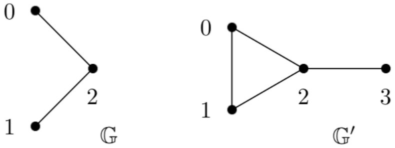
There exists no homomorphism from $\mathbb{G}'$ to $\mathbb{G}$. For suppose h is such a homomorphism. We have $(0,1),(1,2),(2,0) \in E^{\mathbb{G}'}$, so we must have $(h(0),h(1)),(h(1),h(2)),(h(2),h(0)) \in E^{\mathbb{G}}$. However, it is easy to see that there exist no points x,y,z in $\mathbb{G}$ such that $(x,y),(y,z),(z,x) \in E^{\mathbb{G}}$. (Note, however, that there are points x,y in $\mathbb{G}$ such that $(x,y),(y,x) \in E^{\mathbb{G}}$.)

Notes on some further types of models



Let us calculate the number of automorphisms from $\mathbb{G}$ to $\mathbb{G}$. First, there always exists the trivial automorphism. Secondly, it is easy to see that under every automorphism f, we have $f(2) = 2$. Therefore the only remaining possible automorphism is the map $\{(0,1),(1,0),(2,2)\}$. This is indeed an automorphism, so alltogether we have two automorphisms. Thus $\mathbb{G}$ is not rigid.

Notes on some further types of models



Let us identify all partial isomorphisms p with domain $\{1, 2\}$ from $\mathbb{G}$ to $\mathbb{G}'$. First note that $(1, 2), (2, 1) \in E^{\mathbb{G}}$, so we must have

$$(p(1), p(2)), (p(2), p(1)) \in E^{\mathbb{G}'}.$$

Thus there four possible sets $\{p(1), p(2)\}$, the sets

$$\{0, 1\}, \{0, 2\}, \{1, 2\}, \{2, 3\}.$$

For each of the four sets, there are two possible partial automorphisms, e.g., $p = \{(1, 0), (2, 1)\}$ and $p = \{(1, 1), (2, 0)\}$ in the case of the set $\{0, 1\}$. Thus there exist eight partial isomorphisms with domain $\{1, 2\}$.