Finite model theory

5. Logics III

Fixed Point Logics and FVL

We shall next discuss **fixed point logics**. These can be considered to be finitary approximants of FVL. The related theory is not difficult, but it is technically messy at parts. The main objective of this section is to learn to write formulae of the logic LFP (least fixed point logic), i.e., to specify simple properties in LFP. We shall also prove some basic results concerning monotone operators.

Fixed Point Operators

Definition 5.1

Let S be a set. An **operator** on S is a mapping $F : \mathcal{P}(S) \to \mathcal{P}(S)$. The operator F is **monotone** if

$$X \subseteq Y \;\Rightarrow\; F(X) \subseteq F(Y).$$

Example 5.2

Let (S, R) be a model with a binary relation $R \subseteq S \times S$. Define the operator $F : \mathcal{P}(S) \to \mathcal{P}(S)$ such that

$$F(X) = \{u \in S \mid \exists y((u, y) \in R \;\wedge y \in X)\}.$$

Thus, on the input X, the operator outputs those points that connect directly via R to some point in X. This is a monotone operator. For assume $X \subseteq Y$. Let $x \in F(X)$, so $(x, y) \in R$ and $y \in X$ for some y. Therefore $(x, y) \in R$ and $y \in Y$, and thus $x \in F(Y)$. Therefore $F(X) \subseteq F(Y)$.

Fixed Point Operators

Example 5.3

Define $F : \mathcal{P}(S) \to \mathcal{P}(S)$ such that $F(X) = S \setminus X$. This operator is not monotone. For if $S \neq \emptyset$, then $\emptyset \subseteq S$ but $F(\emptyset) \not\subseteq F(S)$.

Let $A \subseteq S$, $A \neq S$. Define the operator $G : \mathcal{P}(S) \to \mathcal{P}(S)$ such that $G(X) = A$ for all $X \in \mathcal{P}(S)$. This is a monotone operator.

Fixed Point Operators

Definition 5.4

Let $F : \mathcal{P}(S) \to \mathcal{P}(S)$ be an operator. A set $X \subseteq S$ is a **fixed point** (or a **fixpoint**) of F if $F(X) = X$. The set X is the **least fixed point** of F, or $lfp(F)$, if X is a fixed point of F and we have $X \subseteq Y$ for all fixed points Y of F.

Fixed Point Operators

Let $F : \mathcal{P}(S) \to \mathcal{P}(S)$ be an operator on a finite set S. Consider the sequence

$$X^0 = \emptyset \quad \text{and} \quad X^{n+1} = F(X^n).$$

The sequence is **inductive** if it is weakly increasing, i.e., $X^n \subseteq X^{n+1}$ for all $n \in \mathbb{N}$.[1] We write

$$X^\infty = \bigcup_{i \geq 0} X^i.$$

[1] A strongly increasing sequence would have $X^n \subseteq X^{n+1}$ and $x^{n+1} \neq X^n$ for all $n \in \mathbb{N}$. However, on a finite set, strongly increasing sequences cannot exist.

Fixed Point Operators

Example 5.5

Let (A, R) be a model where R is a binary relation over A. Define an operator $F : \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$ over $A \times A$ such that

$$F(X) = R \cup (R \circ X),$$

where $R \circ X$ is the relational composition of R and X, i.e.,

$$R \circ X = \{(u, v) \,|\, (u, y) \in R \ \& \ (y, v) \in X \text{ for some } y \in A\}.$$

The operator F is monotone, for suppose that $X \subseteq Y$ are binary relations over A. Then $(R \circ X) \subseteq (R \circ Y)$, so clearly $F(X) \subseteq F(Y)$. Consider the sequence $X^0, X^1, X^2 \ldots$ and its relation to the relation R. We have $X^0 = \emptyset$ and $X^1 = R$. Then we have $X^2 = R \cup (R \circ R)$, and continuing in this fashion, we see that X^∞ is clearly the transitive closure of R.

Fixed Point Operators

Theorem 5.6 (Knaster-Tarski)

1. *Let* $F : \mathcal{P}(S) \to \mathcal{P}(S)$ *be a monotone operator. Then* F *has a least fixed point, and the least fixed point is given by*

$$X = \bigcap \{ Y \mid F(Y) = Y \},$$

in other words, it is the intersection of the fixed points of F*.*

2. *Supposing* S *if finite, the least fixed point of* F *is the set*

$$X^{\infty} = \bigcup_{n \geq 0} X^{n}.$$

Proof. Begins from the next slide...

Fixed Point Operators

Define $\mathbf{D} = \{Y \mid F(Y) \subseteq Y\}$. The set $\mathbf{D}$ is nonempty, as $S \in \mathbf{D}$. Therefore $\bigcap \mathbf{D}$ exists. Our first aim is to show that $\bigcap \mathbf{D}$ is a fixed point of F.

Now, let $Y \in \mathbf{D}$. We clearly have $\bigcap \mathbf{D} \subseteq Y$. Therefore

$$F(\bigcap \mathbf{D}) \subseteq F(Y) \subseteq Y$$

due to the monotonicity of F and definition of $\mathbf{D}$. Therefore, because this holds for an arbitary $Y \in \mathbf{D}$, we have

$$F(\bigcap \mathbf{D}) \subseteq \bigcap \mathbf{D}.$$

Therefore we have $F(F(\bigcap \mathbf{D})) \subseteq F(\bigcap \mathbf{D})$ by the monotonicity of F. Thus $F(\bigcap \mathbf{D}) \in \mathbf{D}$ by the definition of $\mathbf{D}$. Therefore $\bigcap \mathbf{D} \subseteq F(\bigcap \mathbf{D})$. Altogether, we have thus shown that $\bigcap \mathbf{D} = F(\bigcap \mathbf{D})$ (by establishing inclusions in both directions). Therefore $\bigcap \mathbf{D}$ is a fixed point of F.

Fixed Point Operators

As $\bigcap \mathbf{D} = F(\bigcap \mathbf{D})$, the set $\bigcap\{ Y \mid F(Y) = Y \}$ is defined (i.e., it is not an intersection over the empty set). Trivially

$$\{ Y \mid F(Y) = Y \} \subseteq \{ Y \mid F(Y) \subseteq Y \} = \mathbf{D},$$

so we have

$$\bigcap \mathbf{D} = \bigcap\{ Y \mid F(Y) \subseteq Y \} \subseteq \bigcap\{ Y \mid F(Y) = Y \}.$$

As we already proved that $\bigcap \mathbf{D}$ is a fixed point of F, it now suffices to prove that $\bigcap\{ Y \mid F(Y) = Y \} \subseteq \bigcap\{ Y \mid F(Y) \subseteq Y \}$, as then the set $\bigcap\{ Y \mid F(Y) = Y \} = \bigcap \mathbf{D}$ is clearly the least fixed point, being the intersection of all fixed points. The set $\bigcap \mathbf{D}$ being a fixed point, we know that $\bigcap \mathbf{D} = F(\bigcap \mathbf{D})$, so $\bigcap \mathbf{D} \in \{ Y \mid F(Y) = Y \}$. Hence

$$\bigcap\{ Y \mid F(Y) = Y \} \subseteq \bigcap \mathbf{D} = \bigcap\{ Y \mid F(Y) \subseteq Y \}$$

as desired. This proves claim 1 of the theorem.

Fixed Point Operators

We then prove the claim 2 of the theorem, i.e., the claim that if S if finite, the least fixed point of F is the set

$$X^\infty = \bigcup_{n \geq 0} X^n.$$

To see that X^∞ is a fixed point, let us first show that that the sequence $X^0, X^1, X^2 \ldots$ is weakly increasing due to the monotonicity of F. Indeed, we have $X^0 = \emptyset \subseteq X^1$, and if $X^n \subseteq X^{n+1}$, then, by the monotonicity of F, we have

$$X^{n+1} = F(X^n) \subseteq F(X^{n+1}) = X^{n+2}.$$

Therefore, as the set S is finite, the sequence must increase up to some index n such that $X^n = X^{n+1}$, whence

$$X^\infty = X^n = X^{n+1}$$

is indeed a fixed point. Let us then show X^∞ is the least fixed point.

Fixed Point Operators

Now, to show that X^∞ is indeed the least fixed point, suppose that we have $F(Y) = Y$. It now suffices to show that $X^n \subseteq Y$ for all $n \in \mathbb{N}$. To this end, we argue by induction on n.

Now, we have $X^0 = \emptyset \subseteq Y$, so the base case is clear. Suppose then that $X^n \subseteq Y$. Thus, by the monotonicity of F, we have

$$X^{n+1} = F(X^n) \subseteq F(Y) = Y,$$

whence $X^{n+1} \subseteq Y$, as required.$\square$

Fixed Point Logic

Fixed points can be used to define logics that approximate the finite variable logic FVL. This leads to logics that are not infinitary like FVL, but nevertheless can express a variety of interesting and relevant properties—such as c_1-c_2-connectedness—that are not expressible in FO. Here we next define the most prominent of the range of related logics, namely the **least fixed point logic** LFP. The logic is somewhat difficult to understand and involves many issues that need to be taken into account when introducing the system. We begin with some auxiliary definitions.

Fixed Point Logic

Let φ be a formula and P a relation symbol (or a relation variable) of any arity. We say that P **occurs positively** in φ if P is in the scope of an even number of negations.[2] For example P occurs positively in the formula $\exists x(\neg Qx \wedge \neg(Rxy \vee \neg Px))$, being in the scope of two negations, and P occurs negatively in the formula $\exists x \neg(\neg Qx \wedge \neg(Rxy \vee \neg Px))$, being in the scope of three negations.

Of course P can also occur both positively and negatively in a formula.

[2] Being in the scope of a negation means P is somewhere below that negation in the syntax tree (or parse tree) of the formula.

Fixed Point Logic

Let τ be a vocabulary and $\mathfrak{A}$ a τ-model. Let S be an n-ary relation symbol such that $S \notin \tau$. Let $S' \subseteq A^n$ be an n-ary relation over A. We let $\mathfrak{A}[S \mapsto S']$ denote the $(\tau \cup \{S\})$-model that is otherwise as $\mathfrak{A}$ but interprets the extra symbol S as the relation S'. That is, $\mathfrak{A}[S \mapsto S']$ is the expansion of $\mathfrak{A}$ to the vocabulary $\tau \cup \{S\}$ interpreting S as S'.

Let $f : V \to A$ be an assignment, and let $a_1, \dots, a_n \in A$. We let $f[(x_1, \dots, x_n) \mapsto (a_1, \dots, a_n)]$ denote the assignment with the domain $V \cup \{x_1, \dots, x_n\}$ and such that

- $f[(x_1, \dots, x_n) \mapsto (a_1, \dots, a_n)](y) = f(y)$ when $y \notin \{x_1, \dots, x_n\}$.
- $f[(x_1, \dots, x_n) \mapsto (a_1, \dots, a_n)](x_i) = a_i$ for all $x_i \in \{x_1, \dots, x_n\}$.

In other words, $f[(x_1, \dots, x_n) \mapsto (a_1, \dots, a_n)]$ is otherwise as f but maps each x_i to a_i.

Fixed Point Logic

Recall that a formula $\varphi(x, y)$ gives rise to the binary relation $\varphi^{\mathfrak{A}} \subseteq A \times A$ in a model $\mathfrak{A}$, the relation $\varphi^{\mathfrak{A}}$ containing exactly those pairs $(a, b) \in A \times A$ such that $\mathfrak{A}, f[(x, y) \mapsto (a, b)] \models \varphi$. This leads to the following idea.

Let $\mathfrak{A}$ be a finite τ-model with $\tau = \{R\}$ containing one binary relation symbol R. Let φ be the formula $R(x, y) \vee \exists z(R(x, z) \wedge X(z, y))$, where X is a binary relation symbol (or alternatively a binary relation variable) that does not occur in τ. Now, consider the relation $\varphi^{\mathfrak{A}[X \mapsto \emptyset]}$, that is, the relation defined by φ when $\mathfrak{A}$ is expanded by the new relation symbol X interpreted as the empty binary relation $\emptyset$. Clearly $\varphi^{\mathfrak{A}[X \mapsto \emptyset]} = R^{\mathfrak{A}}$. Call $Y^0 = \emptyset$ and $Y^1 = \varphi^{\mathfrak{A}[X \mapsto Y^0]} = R^{\mathfrak{A}}$, and now consider $\varphi^{\mathfrak{A}[X \mapsto Y^1]}$. This is the relation $R^{\mathfrak{A}} \cup (R^{\mathfrak{A}} \circ R^{\mathfrak{A}})$. Call this relation Y^2 and consider $\varphi^{\mathfrak{A}[X \mapsto Y^2]}$. This is the relation $R^{\mathfrak{A}} \cup (R^{\mathfrak{A}} \circ Y^2)$. It is easy to see that continuing in this fashion, we ultimately obtain a fixed point and it is the transitive closure of $R^{\mathfrak{A}}$. Thereby, intuitively, we were able to obtain the transitive closure of a binary relation by 'repeatedly inputting the output relation of the formula φ to the formula itself'.

Fixed Point Logic

Inspired by this example, let us put forward a related general definition that considers formulae φ as operators over A^n.

Let τ be a vocabulary. Let $S \notin \tau$ be an n-ary relation symbol, and let φ be a formula of the vocabulary $\{S\} \cup \tau$. Let $\mathfrak{A}$ be a τ-model and $f : V \to A$ a related assignment. The formula φ gives rise to an operator

$$F_{(\mathfrak{A}, f, \varphi, S, (x_1, \ldots, x_n))} : \mathcal{P}(A^n) \to \mathcal{P}(A^n)$$

defined such that

$$\begin{aligned} & F_{(\mathfrak{A}, f, \varphi, S, (x_1, \ldots, x_n))}(X) \\ & = \{\, (a_1, \ldots, a_n) \in A^n \,|\, \mathfrak{A}[S \mapsto X], f[(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)] \models \varphi \}. \end{aligned}$$

So informally, the output is the collection of tuples $(a_1, \ldots, a_n)$ such that $\varphi(a_1, \ldots, a_n)$ holds when S (which occurs in φ) is interpreted as the input X.

Fixed Point Logic

Let τ be a vocabulary. The set of τ-formulae of **least fixed point logic** LFP is obtained by extending the formula construction rules of first-order logic by the following rule:

Suppose S is an n-ary relation symbol that does not occur in τ and φ is a $(\tau \cup \{S\})$-formula of LFP where all occurrences of S are positive. Let $t_1, \dots, t_n$ be τ-terms. Then

$$[\mathit{lfp}_{((x_1,\dots,x_n),S)}\ \varphi](t_1, \dots, t_n)$$

is a τ-formula of LFP.

Fixed Point Logic

The semantics of LFP is defined by extending the rules of first-order logic such that

$$\mathfrak{A}, f \models [\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi](t_1, \ldots, t_n)$$

holds if and only if $(t_1^{\mathfrak{A},f}, \ldots, t_n^{\mathfrak{A},f})$ belongs to the least fixed point of the operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$.

Note that therefore you should think about $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi]$ as an n-ary relation obtained by iterating the formula φ up to a fixed point. The formula $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi](t_1, \ldots, t_n)$ then is a relational statement stating that the tuple corresponding to $(t_1, \ldots, t_n)$ is in the relation $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi]$. Thus the formula $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi](t_1, \ldots, t_n)$ is almost like an atomic formula, but now the relation symbol (that an atomic formula would have) has been replaced by $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi]$.

Examples of LFP-formulae are given below after we have discussed the proof of Theorem 5.7 below.

Fixed Point Logic

Theorem 5.7

Let φ be a formula of LFP such that the occurrences of the relation symbol S are positive. Let $\mathfrak{A}$ be a finite model and f a related assignment. Then the operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$ is monotone and thereby (due to the Knaster-Tarski Theorem) has a least fixed point.

We omit the full proof of the theorem. However, we sketch the proof for the special case where formulae have no nested fixed points. Formulae without nested fixed points are such that φ in any fixed point subformula $[lfp_{(S,(x_1,\ldots,x_n))}\varphi](t_1,\ldots,t_n)$ is not allowed to contain a subformula of type $[lfp_{(T,(y_1,\ldots,y_m))}\psi](s_1,\ldots,s_m)$. Before the proof, we give some auxiliary definitions.

Fixed Point Logic

Definition 5.8

Let U be a set. We say that an operation $F : \mathcal{P}(U) \to \mathcal{P}(U)$ is **antimonotone** if $X \subseteq Y$ implies that $F(Y) \subseteq F(X)$.

Definition 5.9

An operator $G : \mathcal{P}(U) \to \mathcal{P}(U)$ is called the **complement operator of** $F : \mathcal{P}(U) \to \mathcal{P}(U)$ if $G(X) = U \setminus F(X)$ for all $X \in \mathcal{P}(U)$.

Clearly the complement operator of a monotone operator is antimonotone.

Note, above G is the complement operator **of** F. The general **complement operator** is the operator $C_U : \mathcal{P}(U) \to \mathcal{P}(U)$ such that $C_U(X) = U \setminus X$ for every input $X \in \mathcal{P}(U)$.

Fixed Point Logic

Consider an operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$ which maps from $\mathcal{P}(A^n)$ to $\mathcal{P}(A^n)$. Recall that this operator is defined such that

$$\begin{aligned}&F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}(X)\\&=\{\,(a_1,\ldots,a_n)\in A^n \mid \mathfrak{A}[S\mapsto X], f[(x_1,\ldots,x_n)\mapsto(a_1,\ldots,a_n)]\models\varphi\}.\end{aligned}$$

It is easy to see that $F_{(\mathfrak{A},f,\neg\varphi,S,(x_1,\ldots,x_n))}$ is the complement operator of the operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$.

Proof Sketch. Here sketch the proof of Theorem 5.7 for the case where there are no nested fixed points. The argument is somewhat lengthy, tedious and not too central to our discourse, so the reader can safely skip over the proof sketch.

Fixed Point Logic

Now, recall the restriction in the syntax of LFP that in all formulae of type $[\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\varphi](t_1,\ldots,t_n)$, all occurrences of S in φ must be positive in φ. Therefore it is easy to see that for every subformula ψ of φ, we have that

1. all occurrences of S are positive in ψ

   or

2. all occurrences of S are negative in ψ.

We shall prove by induction on the structure of φ that for all subformulae ψ of φ,

- if all occurrences of S in ψ are positive, then $F_{(\mathfrak{A},f,\psi,S,(x_1,\ldots,x_n))}$ is monotone, and
- if all occurrences of S in ψ are negative then $F_{(\mathfrak{A},f,\psi,S,(x_1,\ldots,x_n))}$ is antimonotone.

Fixed Point Logic

If ψ is an atom $R(t_1, \ldots, t_m)$ or $t_1 = t_2$, it is clear that

$$F_{(\mathfrak{A}, f, \psi, S, (x_1, \ldots, x_n))}(X)$$
$$= \{\, (a_1, \ldots, a_n) \in A^n \,|\, \mathfrak{A}[S \mapsto X], f[(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)] \models \psi \}$$

defines a constant operator that always outputs the same relation, independently of the input relation X. And constant operators are obviously both monotone and antimonotone.

If ψ is an atom $S(t_1, \ldots, t_n)$, then clearly

$$F_{(\mathfrak{A}, f, \psi, S, (x_1, \ldots, x_n))}(X)$$
$$= \{\, (a_1, \ldots, a_n) \in A^n \,|\, \mathfrak{A}[S \mapsto X], f[(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)] \models \psi \}$$

defines a monotone operator.

Fixed Point Logic

Now assume ψ is $\neg\chi$. By the induction hypothesis,

$$
\begin{aligned}
& F_{(\mathfrak{A}, f, \chi, S, (x_1, \ldots, x_n))}(X) \\
& = \{ (a_1, \ldots, a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)] \models \chi \}
\end{aligned}
$$

is monotone if all occurrences of S are positive in χ, and antimonotone if negative. Now, if $F_{(\mathfrak{A}, f, \chi, S, (x_1, \ldots, x_n))}$ is monotone, then clearly

$$
\begin{aligned}
& F_{(\mathfrak{A}, f, \neg\chi, S, (x_1, \ldots, x_n))}(X) \\
& = \{ (a_1, \ldots, a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)] \models \neg\chi \}
\end{aligned}
$$

is the complement operation of $F_{(\mathfrak{A}, f, \chi, S, (x_1, \ldots, x_n))}$ and thereby antimonotone, as required. Similarly, if $F_{(\mathfrak{A}, f, \chi, S, (x_1, \ldots, x_n))}$ is antimonotone, then $F_{(\mathfrak{A}, f, \neg\chi, S, (x_1, \ldots, x_n))}$ is monotone, as required.

Fixed Point Logic

Consider then the case where $\psi = \psi_1 \wedge \psi_2$. We have two cases:

1. The occurrences of S in ψ are all positive, and thus this holds also for ψ_1 and ψ_2. Therefore $F_{(\mathfrak{A}, f, \psi_1, S, (x_1, \ldots, x_n))}$ and $F_{(\mathfrak{A}, f, \psi_2, S, (x_1, \ldots, x_n))}$ are both monotone by the induction hypothesis.
2. The occurrences of S in ψ are all negative, and thus $F_{(\mathfrak{A}, f, \psi_1, S, (x_1, \ldots, x_n))}$ and $F_{(\mathfrak{A}, f, \psi_2, S, (x_1, \ldots, x_n))}$ are both antimonotone by the induction hypothesis.

We first consider case 1.

Fixed Point Logic

Suppose $X \subseteq Y$. We must show that

$$F_{(\mathfrak{A},f,\psi_1\wedge\psi_2,S,(x_1,\ldots,x_n))}(X) \subseteq F_{(\mathfrak{A},f,\psi_1\wedge\psi_2,S,(x_1,\ldots,x_n))}(Y),$$

i.e., that

$$\{ (a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_1 \wedge \psi_2\}$$
$$\subseteq \{(a_1,\ldots,a_n) \in A^n | \mathfrak{A}[S \mapsto Y], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_1 \wedge \psi_2\}.$$

This amounts to showing that

$$\{ (a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_1\}$$
$$\cap \{ (a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_2\}$$
$$\subseteq$$
$$\{ (a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto Y], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_1\}$$
$$\cap \{ (a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto Y], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \psi_2\}.$$

In other words, we must show that ...

Fixed Point Logics

$$F_{(\mathfrak{A},f,\psi_1,S,(x_1,\dots,x_n))}(X) \cap F_{(\mathfrak{A},f,\psi_2,S,(x_1,\dots,x_n))}(X)$$
$$\subseteq$$
$$F_{(\mathfrak{A},f,\psi_1,S,(x_1,\dots,x_n))}(Y) \cap F_{(\mathfrak{A},f,\psi_2,S,(x_1,\dots,x_n))}(Y).$$

By the induction hypothesis, the assumption $X \subseteq Y$ implies that we have

1. $F_{(\mathfrak{A},f,\psi_1,S,(x_1,\dots,x_n))}(X) \subseteq F_{(\mathfrak{A},f,\psi_1,S,(x_1,\dots,x_n))}(Y)$ and
2. $F_{(\mathfrak{A},f,\psi_2,S,(x_1,\dots,x_n))}(X) \subseteq F_{(\mathfrak{A},f,\psi_2,S,(x_1,\dots,x_n))}(Y)$,

and thus the case where both $F_{(\mathfrak{A},f,\psi_1,S,(x_1,\dots,x_n))}$ and $F_{(\mathfrak{A},f,\psi_2,S,(x_1,\dots,x_n))}$ are monotone is clear. The case where both of these operators are antimonotone is symmetric.

Fixed Point Logic

Now let $\psi = \exists x \chi$. Suppose that the occurrences of S in χ are all positive, so $F_{(\mathfrak{A}, f, \chi, S, (x_1, \ldots, x_n))}$ is monotone by the induction hypothesis. We assume $x \notin \{x_1, \ldots, x_n\}$ (the assumption is fine as we can always rename the occurrences of x in χ if necessary). We must show that...

Fixed Point Logic

$$F_{(\mathfrak{A},f,\exists x\chi,S,(x_1,\ldots,x_n))}(X) \subseteq F_{(\mathfrak{A},f,\exists x\chi,S,(x_1,\ldots,x_n))}(Y), \text{ i.e., that}$$

$$\{\,(a_1,\ldots,a_n) \in A^n \mid \mathfrak{A}[S \mapsto X], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \exists x\chi\}$$
$$\subseteq \{(a_1,\ldots,a_n) \in A^n | \mathfrak{A}[S \mapsto Y], f[(x_1,\ldots,x_n) \mapsto (a_1,\ldots,a_n)] \models \exists x\chi\}.$$

We have

$$\begin{aligned}
& (a_1,\ldots,a_n) \in F_{(\mathfrak{A},f,\exists x\chi,S,(x_1,\ldots,x_n))}(X) \\
\Rightarrow \\
& (a_1,\ldots,a_n) \in F_{(\mathfrak{A},f[x\mapsto a],\chi,S,(x_1,\ldots,x_n))}(X) \text{ for some } a \in A \\
\overset{ind.hypot.}{\Rightarrow} \\
& (a_1,\ldots,a_n) \in F_{(\mathfrak{A},f[x\mapsto a],\chi,S,(x_1,\ldots,x_n))}(Y) \text{ for some } a \in A \\
\Rightarrow \\
& (a_1,\ldots,a_n) \in F_{(\mathfrak{A},f,\exists x\chi,S,(x_1,\ldots,x_n))}(Y),
\end{aligned}$$

as required. The case for antimonotone ${}_{(\mathfrak{A},f,\chi,S,(x_1,\ldots,x_n))}$ is symmetric.$\square$

Fixed Point Logic

Write a formula of LFP that defines the property of c_1-c_2-connectedness over the class of finite $\{R, c_1, c_2\}$-models.

Fixed Point Logic

$$c_1 = c_2 \ \vee \ [\mathit{lfp}_{(S,(x,y))} Rxy \vee \exists z(Rxz \wedge Szy)](c_1, c_2)$$

Fixed Point Logic

From the example above, we see that LFP indeed relates quite closely to FVL. We shall return to this issue later on below.

Fixed Point Logic

Consider the following game that is played on finite structures (V, R, c) where R is a binary relation and c a constant. The game has two players, I and II. At the first round, player I chooses some v_1 such that $(c, v_1) \in R$ and player II responds by choosing some u_1 such that $(v_1, u_1) \in R$. At every subsequent round $i+1$, the player I chooses a node v_{i+1} such that $(u_i, v_{i+1}) \in R$ and player II responds by choosing some u_{i+1} such that $(v_{i+1}, u_{i+1}) \in R$. The player who cannot make a legal move loses the game, and then the other player wins. If the game continues forever, neither of the players wins the game.

Consider the formula $\varphi(x) = \forall y(Rxy \to \exists z(Ryz \wedge S(z))$. Describe the set $A \subseteq V$ of elements a such that

$$(V, R, c), f[y \mapsto a] \models [lfp_{(S,x)}\varphi(x)](y).$$

FIxed Point Logic

The set A is the set of nodes from where player II has a winning strategy in the game if the player I has the turn to make a move. Thus we have $(V, R, c) \models [lfp_{(S,x)}\varphi(x)](c)$ if the player II has a winning strategy in the game.

Fixed Point Logic

The requirement that the occurrences of S in formulae

$$[lfp_{(S,(x_1,\ldots,x_n))}\varphi](t_1,\ldots,t_n)$$

are positive makes the related operators $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$ monotone and thereby results in the existence of fixed points by Theorem 5.7. However, the requirement of positive occurrences can be considered a theoretical weakness and nuisance. Thus there exist further fixed point logics in finite model theory we shall discuss next. The logics are very similar to LFP but avoid the requirement of positive occurrences. We begin with some auxiliary definitions.

Fixed Point Logic

Definition 5.10

Let $F : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ be an operator. The operator is **inflationary** if we have $X \subseteq F(X)$ for all $X \in \mathcal{P}(U)$.

If U is a finite set and $F : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ inflationary, then clearly the sequence $X^0, X_1, X^2 \ldots$ (where $X^0 = \emptyset$ and $F^{n+1}(X) = F(F^n(X))$) is inductive and thus F reaches a fixed point such that $X^n = X^{n+1}$.

Definition 5.11

Let $F : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ be an operator. We let $F_{infl} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ denote the corresponding inflationary operator defined such that $F_{infl}(X) = X \cup F(X)$ for all $X \in \mathcal{P}(U)$.

Fixed Point Logic

Definition 5.12

Let $F : \mathcal{P}(U) \to \mathcal{P}(U)$ be an operator on a finite set U. Let X^0, X^1, X^2 be the inductive sequence defined by the operator F_{infl}. As pointed out above, since U if finite, the sequence reaches a fixed point, i.e., $X^n = X^{n+1}$ for some n. That set X^n is called the **inflationary fixed point** of F. We let $ifp(F)$ denote the inflationary fixed point of F.

Fixed Point Logic

Definition 5.13

Let $F : \mathcal{P}(U) \to \mathcal{P}(U)$ be an operator on a finite set U. Let $X^0, X^1, X^2, \ldots$ be the corresponding sequence with $X^0 = \emptyset$ and $F^{n+1}(X) = F(F(X^n))$. The **partial fixed point** of F is the set $Y \in \mathcal{P}(U)$ defined as follows.

- Suppose there exists an n such that $X^n = X^{n+1}$. Then the partial fixed point of F is the set X^n.
- If no such n exists, the partial fixed point of F is $\emptyset$.

We denote the partial fixed point of F by $pfp(F)$.

Fixed Point Logic

Let τ be a vocabulary. The set of τ-formulae of **inflationary fixed point logic** IFP is defined by extending the formula construction rules of first-order logic by the following rule.

Suppose S is an n-ary relation symbol that does not occur in τ and φ is a $(\tau \cup \{S\})$-formula of IFP. Let $t_1, \ldots, t_n$ be τ-terms. Then

$$[ifp_{((x_1,\ldots,x_n),S)}\ \varphi](t_1, \ldots, t_n)$$

is a τ-formula of IFP.

Notice that we do not require that the occurrences of S should be positive in φ.

Fixed Point Logic

The semantics of inflationary fixed point logic IFP is defined by extending the rules of first-order logic such that

$$\mathfrak{A}, f \models [\mathit{lfp}_{(S,(x_1,\ldots,x_n))}\ \varphi](t_1,\ldots,t_n)$$

holds if and only if $(t_1^{\mathfrak{A},f},\ldots,t_n^{\mathfrak{A},f})$ belongs to the inflationary fixed point of the operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$.

Fixed Point Logic

Let τ be a vocabulary. The set of τ-formulae of **partial fixed point logic** PFP is defined by extending the formula construction rules of first-order logic by the following rule.

Suppose S is an n-ary relation symbol that does not occur in τ and φ is a $(\tau \cup \{S\})$-formula of PFP. Let $t_1, \ldots, t_n$ be τ-terms. Then

$$[pfp_{((x_1,\ldots,x_n),S)}\ \varphi](t_1, \ldots, t_n)$$

is a τ-formula of PFP.

Notice that like in the case of IFP, we do not require that the occurrences of S should be positive in φ.

Fixed Point Logic

The semantics of partial fixed point logic PFP is defined by extending the rules of first-order logic such that

$$\mathfrak{A}, f \models [pfp_{(S,(x_1,\ldots,x_n))}\ \varphi](t_1,\ldots,t_n)$$

holds if and only if $(t_1^{\mathfrak{A},f},\ldots,t_n^{\mathfrak{A},f})$ belongs to the partial fixed point of the operator $F_{(\mathfrak{A},f,\varphi,S,(x_1,\ldots,x_n))}$.

Fixed Point Logic

The formula $[lfp_{(S,(x,y))}\ Rxy \vee \exists z(Rxz \wedge Szy)](x,y)$ defines the transitive closure of the relation R.

1. $\varphi^0(x,y)$ be the formula $\neg x = x \wedge \neg y = y$ (which is never satisfied and thus defines the empty binary relation over any model).
2. Suppose we have defined $\varphi^n(x,y)$. We would like to define $\varphi^{n+1}(x,y)$ to be the formula obtained from $Rxy \vee \exists z(Rxz \wedge Szy)$ by replacing the atom Szy by the formula $\varphi^n(z,y)$.This almost works, but now $\varphi^1(z,y)$ would become $Rzy \vee \exists z(Rzz \wedge \varphi^0(z,y))$. Thus we define $\varphi^{n+1}(x,y)$ to be the formula $Rxy \vee \exists z(Rxz \wedge \exists x(x = z \wedge \varphi^n(x,y)))$, which fixes the variable capture problem. *So, to summarize, $\varphi^{n+1}(x,y)$ is obtained by replacing Sxy in $Rxy \vee \exists z(Rxz \wedge \exists x(x = z \wedge Sxy))$ by $\varphi^n(x,y)$.*

It is easy to see that the infinitary formula

$$\bigvee_{n \in \mathbb{N}} \varphi^n$$

of the finite variable logic FVL is equivalent to the LFP-formula $[lfp_{(S,(x,y))}\ Rxy \vee \exists z(Rxz \wedge Szy)](x,y)$.

Fixed Point Logic

It is not difficult to see how to extend this scheme so that every formula of LFP translates into FVL, but we omit the full proof here.

Theorem 5.14

Every formula of LFP translates to an equivalent formula of FVL.

Thereby we can use tools of FVL (namely, the pebble game PG') to prove undefinability results that concern the least fixed point logic.