Fourier Methods

Lecture Notes

1 Introduction

In these notes we give an overview of the basic results of Fourier analysis important from the point of view of applications. In particular, we introduce the reader to *Fourier series* and the *Fourier transform*.

Fourier series provide a way to describe periodic functions by an infinite sum of trigonometric functions. For example, the Fourier series of the periodic function

$$f(x) = \begin{cases} 1 & \text{if} \quad n\pi < x \leq (n+1)\pi \text{ and } n \text{ is odd} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

is

$$\frac{1}{2} - \frac{2}{\pi}\Big(\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots \Big).$$

In a typical application, it is possible to throw away all but some small finite number of the terms of the infinite Fourier series, and the obtained finite sum of trigonometric terms ***gives a very good approximation of the original function.*** The point of doing this is that the finite sum is typically ***a lot simpler*** than the original function. Indeed, the above specification of the function $f(x)$ is quite complex and very difficult to use in practical calculations. Trigonometric functions are well understood and generally behave very nicely, thus enabling smooth use in applications and also in more theoretical studies.

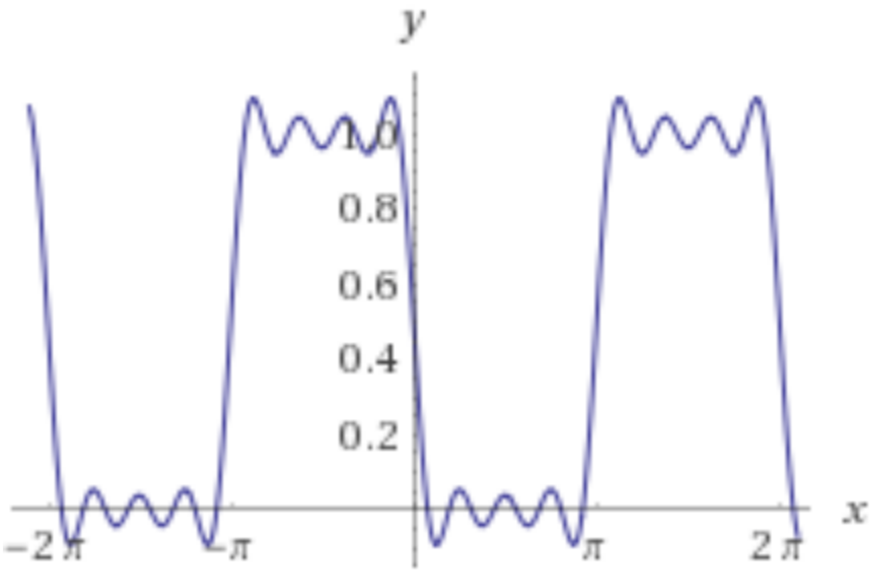

Figure 1: The figure shows the graph of the Fourier series of the above defined function $f(x)$ with all the terms up to $\frac{\sin(7x)}{7}$ included. This is already pretty close to the sawtooth pattern defined by $f(x)$.

Another perspective to the usefulness of Fourier series comes from the analysis of signals. A signal can typically be represented by a periodic function. In applications involving the modification of complicated sound waves, it is often desirable to, inter alia, get rid of some disturbing noises while keeping most of the soundwave intact. Here we can first decompose the complicated sound wave to its Fourier series and then find the problematic harmonic components and remove them.

Indeed, a Fourier series of a periodic function $f(x)$ provides a fundamental ***decomposition*** of $f(x)$ into trigonometric components of the form

$$a_n \sin(n\omega x) \quad \text{and} \quad b_n \cos(n\omega x)$$

for $n \in \mathbb{N}$. These components can be thought of as the fundamental ***building blocks*** of $f(x)$. Modifying the building blocks, we can modify $f(x)$ in a powerful way.

It is worth noting that the notion of a fundamental building block is omnipresent in mathematics. In linear algebra, the basis vectors of a vector space can be thought of as the fundamental building blocks with which everything is built. In number theory, natural numbers decompose into their prime number representations, so primes are the fundamental building blocks there. In logic, all formulae can be build from the so-called *types* using only the disjunction operation. And the list goes on and on. Modifying a complicated object by first decomposing it into a representation in terms of its fundamental building blocks is advantageous, as it is often easiest to modify the object by modifying its simple building blocks.

There are of course other ways of representing functions by infinite series, such as the Maclaurin and Taylor series. The reader is probably already familiar with these. For example the the Maclaurin series of $\sin(x)$ is

$$\sin\big(x\big) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{\big(-1\big)^n}{\big(2n+1\big)!}x^{2n+1}.$$

The Maclaurin expansion is not given in terms of trigonometric functions but as a combination of the basis functions

$$1,\ x,\ x^2,\ x^3, \ldots$$

In Maclaurin series, the function to be expanded ought to be infinitely differentiable. Fourier series have different constraints and can often be much more flexibly obtained.

While Fourier series are used to represent a periodic function by a discrete sum of trigonometric terms, or alternatively, a sum of complex exponentials, the *Fourier transform* is used to represent a general, possibly nonperiodic function as a superposition of complex exponentials.

As this course is directed towards applications, some mathematical details must be omitted in the way typical to applied mathematics courses. However, for the reader interested in a more advanced treatment of the topic, there are numerous books discussing the topic in varying levels of detail. In relation to this, it is worth noting that the definitions of Fourier analysis typically differ slightly from one source to another, but the main lines of approach are of course the same.

2 Preliminaries

2.1 Periods and circular frequencies

Recall that the **domain** of a function f is the set D such that $f(d)$ exists. For example, the domain of $\sin(t)$ is $\mathbb{R}$ and the domain of $\frac{\sin(t)}{t}$ is $\mathbb{R}$ with 0 excluded.

A function f with domain $\mathbb{R}$ is periodic if there exists a positive real number T such that

$$f(t+T) = f(t)$$

for all real numbers t. The number T is a **period** of f.

A T-periodic function automatically has several periods, since

$$f(t+mT) = t$$

holds for any mT, where m is non-negative integer. The shortest period of a periodic function—if such a shortest period exists—is called the **fundamental period** of the function.

Example 2.1. The fundamental period of $\sin(t)$ is 2π, which means that $\sin(t)$ repeats its value after each interval of 2π but no sooner. Also 4π is a period of $\sin(t)$ (cf. Figure 2), and so is every number $m2\pi$ for every positive integer m. $\square$

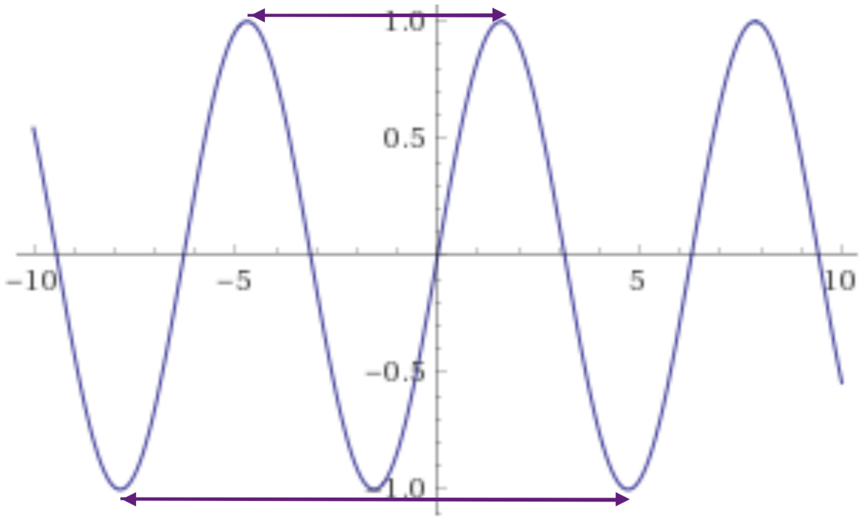

Figure 2: The figure shows $\sin(x)$ plotted from -10 to 10. Both the upper and lower arrow span a possible period of $\sin(x)$; the upper one spans a single fundamental period (i.e., a distance of 2π) and the lower one a period 4π that amounts to two fundamental periods.

Example 2.2. Let us find the fundamental period of $f(t) = \sin(\omega t + \varphi)$, where $\omega > 0$. Let T denote the fundamental period of the function. We have

$$\sin(\omega t + \varphi) = f(t) = f(t + T) = \sin(\omega(t + T) + \varphi) = \sin(\omega t + \omega T + \varphi),$$

and thus

$$\sin(\omega t + \varphi) = \sin(\omega t + \varphi + \omega T).$$

Thus, as the sine function has fundamental period 2π, we must have $\omega T = n2\pi$ for some n. Thus

$$T = \frac{n2\pi}{\omega}.$$

As T is the *smallest* possible positive period, we must have $n = 1$. Therefore

$$T = \frac{2\pi}{\omega}.$$

□

Here we considered the sine function, but cosine can be analysed similarly, so we conclude that the following holds.

> Let $\omega > 0$ and let A be a non-zero constant. The fundamental period of both $A\sin(\omega t + \varphi)$ and $A\cos(\omega t + \varphi)$ is $\dfrac{2\pi}{\omega}$.

Example 2.3. Find a period of $2\cos(\frac{3}{2}t + 9) + \sin(3t)$.

Now, $2\cos(\frac{3}{2}t + 9)$ has the fundamental period

$$\frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$$

and $\sin(3t)$ has the fundamental period

$$\frac{2\pi}{3}.$$

Thus they both have period $\frac{4\pi}{3}$, as $2 \cdot \frac{2\pi}{3} = \frac{4\pi}{3}$. $\square$

We defined periodic functions so that they should have all of $\mathbb{R}$ as their domain. However, in Fourier analysis we often study functions that have *no* specified values at some isolated points. This is not an issue from the point of view of the theory we shall develop. Indeed, we could always *give an arbitrary value* to the functions in the problematic isolated points or *ignore* their behaviour there. This is justified, as it turns out that the Fourier series of a function f is not affected at all if we change the value of f in some isolated point t. (This is ultimately due to the fact that the value of an integral is not changed if the function is changed at some isolated point). Therefore it makes sense to ignore the isolated points, as we can thereby avoid unnecessary mathematical clumsiness, and the applications we are interested in are not affected.

2.1.1 Frequency

Informally, the *frequency* of a periodic function is a measure of how fast the function repeats itself. Consider a periodic function f with the fundamental period T. The **fundamental frequency** of f is defined to be $\frac{1}{T}$. Let g be a periodic function with a period T (which is not necessarily its fundamental period). The **frequency** of g **with respect to** T is defined as $\frac{1}{T}$.

Sometimes the word frequency is used without specifying whether the fundamental frequency is meant or not. Then it should always be sufficiently clear from the context (or irrelevant) what exactly is meant.

Example 2.4. The frequency of $\sin(t)$ with respect to the fundamental period $T = 2\pi$ is $\frac{1}{T} = \frac{1}{2\pi}$, which is about 0.159. The frequency of $\sin(2t)$ with respect to its fundamental period π is $\frac{1}{\pi}$, which is twice the fundamental frequency of $\sin(t)$. Similarly, the frequency of $\sin(3t)$ with respect to its fundamental period $\frac{2\pi}{3}$ is

$$\frac{1}{\frac{2\pi}{3}} = \frac{3}{2\pi},$$

three times the fundamental frequency of $\sin(t)$.

The **fundamental circular frequency** ω of a function f with fundamental period T is defined to be

$$\omega = \frac{2\pi}{T}.$$

Let g be a function with period T, which is not necessarily the fundamental period. The **circular frequency with respect to** T of g is

$$\omega = \frac{2\pi}{T}.$$

Note that circular frequency is obtained from the frequency simply by multiplying by 2π. Intuitively, circular frequency tells us—like frequency—how fast the function repeats itself.

Example 2.5. The function $\sin(t)$ has the fundamental frequency $\frac{1}{2\pi}$ and the fundamental circular frequency $2\pi\frac{1}{2\pi} = 1$. This means that the function repeats itself once within 2π. The function $\sin(\omega t + \varphi)$ has the fundamental circular frequency

$$2\pi \ \cdot \ \frac{1}{\frac{2\pi}{\omega}} = \omega.$$

Thus, for a positive integer n, the function $\sin(nt+\varphi)$ repeats or *peaks* exactly n times within the interval 2π. Thus the fundamental circular frequency of $\sin(\omega t + \varphi)$ tells us exactly the number of times the function *peaks* or *repeats* itself within the interval 2π. $\square$

In calculations of application oriented Fourier analysis, we make constant use of some period T and the related circular frequency $\omega = \frac{2\pi}{T}$. In the standard calculations, it makes no difference whether T and $\omega = \frac{2\pi}{T}$ are fundamental or not.

2.2 Some properties of functions

Definition 2.6. The **left-hand limit** of a function f at $t = a$ is the limit of f at a as t approaches a from below, that is, t is smaller than a and grows closer to a. We denote it by

$$f(a^-) = \lim_{t \to a^-} f(t) = \lim_{h \to 0^+} f(a - h).$$

The **right-hand limit** at $t = a$ is the limit of f at a as t approaches a from above, that is, t is greater than a and decreases towards a. We denote it by

$$f(a^+) = \lim_{t \to a^+} f(t) = \lim_{h \to 0^+} f(a + h).$$

We refer to the left-hand limit and right-hand limit collectively by the term **one-sided limit**.

Example 2.7. The **Heaviside step function** $H(t)$ is defined such that

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

We have $H(0^-) = 0$ and $H(0^+) = 1$. Thus $H(0^-) \neq H(0^+)$. We note that H does not have a value at 0, but this is unrelated to the one-sided limits of H being different. Indeed, define a function g such that

$$g(t) = \begin{cases} H(t) & \text{if } t \neq 0 \\ 6 & \text{if } t = 0. \end{cases}$$

Now $g(0^-) = H(0^-) = 0$ and $g(0^+) = H(0^+) = 1$, while g does have a value at 0. It is also easy to invent functions f such that $f(a^-) = f(a^+) \neq f(a)$ at some point a, that is, the one-sided limits are the same but differ from the value of the function. For example

$$f(t) = \begin{cases} t^2 & \text{if } t \neq 0 \\ 6 & \text{if } t = 0 \end{cases}$$

is such a function. $\square$

We note that sometimes the heaviside function is defined to obtain a value also at 0.

Definition 2.8. Let f be a function that has a value on every point of an interval $[a, b]$. The function f is **piecewise continuous** in the interval $[a, b]$ if

1. $f(a^+)$ and $f(b^-)$ exist (and are not ∞ or $-\infty$),
2. $f(c^-)$ and $f(c^+)$ exist for all $c \in (a, b)$ (and are not ∞ or $-\infty$),
3. $f(c^+) = f(c) = f(c^-)$ for all $c \in (a, b)$ with the possible exception of finitely many $c \in (a, b)$.

The points where the left-hand limits and right-hand limits are not equal are called **points of discontinuity**.

Informally, piecewise continuity of f on $[a, b]$ means that $[a, b]$ can be broken into a finite number of subintervals (u, v) so that

- f is continuous on these subintervals.
- f has a finite limit at the endpoints of each subinterval.

Example 2.9. Define the **sign function** $\mathrm{sgn}(t)$ (also known as the **signum function**) as a function with domain $\mathbb{R}$ such that

$$\mathrm{sgn}(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0. \end{cases}$$

This function is clearly piecewise continous on every interval $[-a, a]$ despite the point of discontinuity 0.

Example 2.10. Let us investigate whether the function

$$f(t) = \begin{cases} \dfrac{1}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

is piecewise continuous in the interval $[-1, 1]$. We observe that at the point 0, neither $f(0^-)$ nor $f(0^+)$ exist as a finite value. Thus f is not piecewise continuous in $[1, -1]$. $\square$

It can be shown that functions f that are piecewise continous in $[a, b]$ are also *bounded* in $[a, b]$. This means that there exist bounds $l \in \mathbb{R}$ and $L \in \mathbb{R}$ such that

$$l < f(c) < L$$

for all $c \in [a, b]$.

Theorem 2.11. *Let f be a periodic function with period T, and suppose f is piecewise continuous on $[0, T]$. Then*

$$\int_d^{d+T} f(t)\, dt = \int_0^T f(t)\, dt.$$

Proof. The existence of the integral

$$\int_0^T f(t)\, dt$$

is clear since we can divide the interval into a finite number of subintervals where f is continuous and then perform a piecewise intergration on the continuous parts. Similarly, the existence of

$$\int_d^{d+T} f(t)\, dt$$

is clear, as again we can divide the interval into a finite number of pieces and perform piecewise integration. Remember that a discontinuity at the end of an interval of integration does not affect the integral.

We now split $\int\limits_{d}^{d+T} f(t)\,dt$ into three integrals as follows:

$$\int_d^{d+T} f\big(t\big)\,dt = \int_d^0 f\big(t\big)\,dt + \int_0^T f\big(t\big)\,dt + \int_T^{d+T} f\big(t\big)\,dt.$$

We substitute $x = t - T$ to $\int_T^{d+T} f\big(t\big)\,dt$ and thus observe that

$$\begin{aligned}\int_d^{d+T} f\big(t\big)\,dt &= \int_d^0 f\big(t\big)\,dt + \int_0^T f\big(t\big)\,dt + \int_0^d f\big(x+T\big)\,dx \\ &= -\int_0^d f\big(t\big)\,dt + \int_0^T f\big(t\big)\,dt + \int_0^d f\big(x\big)\,dx \quad \text{(Recall } f \text{ is periodic.)} \\ &= \int_0^T f\big(t\big)\,dt.\end{aligned}$$

$\square$

The point of this theorem is that when integrating over a period, we can modify the interval of integration freely, as long as the length of the interval is not altered. This can help a lot in practical calculations.

Before finishing the section of with an example, we give the following definition that will be useful later on.

Definition 2.12. We say that a function f is n times continuously differentiable if the nth derivative $f^{(n)}$ exists and is continuous.

We now demonstrate how to integrate piecewise continuous functions. The point is that when integrating over an interval $[a, b]$ where a function is piecewise continuous, we can chop $[a, b]$ at the points of discontinuity and integrate the finitely many continuous parts separately. Note: it is safe to skip over the following example on the first reading and return to it later on.

Example 2.13. Let us define the function

$$f(t) = \begin{cases} 1 & \text{when } \ 2k < t < \big(2k+1\big) \text{ holds for some } k \in \mathbb{Z} \\ 5 & \text{when } \ t = k \text{ holds for some } k \in \mathbb{Z} \\ 2 & \text{when } \ \big(2k-1\big) < t < 2k \text{ holds for some } k \in \mathbb{Z}. \end{cases}$$

What is the fundamental period T and the related circular frequency ω? Evaluate the following definite integrals

$$a_n = \frac{2}{T}\int_0^T f\big(t\big)\cos\big(n\omega t\big)\,dt \qquad \text{and} \qquad b_n = \frac{2}{T}\int_0^T f\big(t\big)\sin\big(n\omega t\big)\,dt,$$

where $n = 0, 1, 2, \ldots$

Solution: The fundamental period is $T = 2$ and the related circular frequency is $\omega = 2\pi/T = \pi$. We have

$$\begin{aligned} a_0 &= \int_0^2 f(t)\,dt \\ &= \int_0^1 1\,dt + \int_1^2 2\,dt = 3. \end{aligned}$$

and for $n > 0$

$$\begin{aligned} a_n &= \int_0^2 f(t)\cos(n\pi t)\,dt \\ &= \int_0^1 \cos(n\pi t)\,dt + \int_1^2 2\cos(n\pi t)\,dt \\ &= \frac{1}{n\pi}\left(\sin(n\pi\cdot 1) - \sin(n\pi\cdot 0)\right) + \frac{2}{n\pi}\left(\sin(n\pi\cdot 2) - \sin(n\pi\cdot 1)\right) = 0 \end{aligned}$$

and

$$
\begin{aligned}
b_n &= \int_0^2 f(t)\sin(n\pi t)\,dt \\
&= \int_0^1 \sin(n\pi t)\,dt + \int_1^2 2\sin(n\pi t)\,dt \\
&= -\frac{1}{n\pi}\left(\cos(n\pi\cdot 1) - \cos(n\pi\cdot 0)\right) - \frac{2}{n\pi}\left(\cos(n\pi\cdot 2) - \cos(n\pi\cdot 1)\right) \\
&= \frac{-1+(-1)^n}{n\pi}.
\end{aligned}
$$

Note that the value 5 of the function f at the points of discontinuity $t = k$ had no effect on the integrals. We could redefine the value 5 arbitrarily, and the same integrals would be obtained.

2.2.1 Even and odd functions

A function is **even** iff

$$f(-t) \;= f(t)$$

for all t in the domain of f. This means that f is symmetric with respect to the y-axis. A function is **odd** if

$$f(-t) = -f(t).$$

for all t in the domain of f. This means that f is symmetric with respect to the origin.

Example 2.14. The function $\cos(t)$ is an example of an even function. Also for example

$$t^2, \quad t^4, \quad |t|$$

are even functions. Examples of odd functions include, for example,

$$\sin(t), \quad t, \quad t^3.$$

Also the signum function $\operatorname{sgn}(t)$ (from Example 2.9) is odd.

The following diagram is useful when determining whether a function is even or odd.

$$\boxed{\begin{array}{lcccc} \text{even} \cdot \text{even} & & & = & \text{even} \\ \text{odd} \cdot \text{even} & = & \text{even} \cdot \text{odd} & = & \text{odd} \\ \text{odd} \cdot \text{odd} & & & = & \text{even} \end{array}}$$

These rules are very easy to prove. We verify one of them:

Example 2.15. Let us show that the product of two odd functions is even. Let f and g be odd. Let h be the function such that $h(t) = f(t) \cdot g(t)$. Therefore

$$h(-t) = f(-t) \cdot g(-t) = (-f(t)) \cdot (-g(t)) = f(t) \cdot g(t) = h(t). \square$$

Note carefully that these rules for multiplying *functions* are not identical to the corresponding rules for multiplying *numbers*: the product of two odd *numbers* is odd.

Also the summation of functions can be associated with similar rules:

$$\boxed{\begin{array}{lcl} \text{even} + \text{even} & = & \text{even} \\ \text{odd} + \text{odd} & = & \text{odd.} \end{array}}$$

These are also easy to verify. Note that the sum of an even and an odd function does not have to be even or odd. This is, likewise, easy to show.

Knowing whether a function is even or odd can simplify integrals a great deal. Let f_{odd} denote an odd function. Let us integrate f_{odd} over the *symmetric* interval $[-c, c]$, where $c > 0$. We have:

$$\begin{aligned}
\int_{-c}^{c} f_{odd}(t)\,dt &= \int_{-c}^{0} f_{odd}(t)\,dt + \int_{0}^{c} f_{odd}(t)\,dt \\
&= \int_{c}^{0} f_{odd}(-t)\,(-dt) + \int_{0}^{c} f_{odd}(t)\,dt \quad \text{(by a change of variables)} \\
&= \int_{c}^{0} -f_{odd}(t)\,(-dt) + \int_{0}^{c} f_{odd}(t)\,dt \quad \text{(since } f_{odd} \text{ is odd)} \\
&= \int_{0}^{c} f_{odd}(t)\,(-dt) + \int_{0}^{c} f_{odd}(t)\,dt \\
&= -\int_{0}^{c} f_{odd}(t)\,dt + \int_{0}^{c} f_{odd}(t)\,dt \\
&= 0.
\end{aligned}$$

Analogously, if g_{even} is an even function, we get

$$\begin{aligned}
\int_{-c}^{c} g_{even}(t)\,dt &= \int_{-c}^{0} g_{even}(t)\,dt + \int_{0}^{c} g_{even}(t)\,dt \\
&= \int_{c}^{0} g_{even}(-t)\,(-dt) + \int_{0}^{c} g_{even}(t)\,dt \quad \text{(by a change of variables)} \\
&= \int_{c}^{0} g_{even}(t)\,(-dt) + \int_{0}^{c} g_{even}(t)\,dt \quad \text{(since } g_{even} \text{ is even)} \\
&= \int_{0}^{c} g_{even}(t)\,dt + \int_{0}^{c} g_{even}(t)\,dt \\
&= 2\int_{0}^{c} g_{even}(t)\,dt.
\end{aligned}$$

To summarize,

$$\boxed{\begin{aligned} \int_{-c}^{c} f(t)\,dt &= 0 && \text{for odd } f && (1) \\ \int_{-c}^{c} f(t)\,dt &= 2\int_{0}^{c} f(t)\,dt && \text{for even } f && (2) \end{aligned}}$$

It is also useful to know that when differentiating an even function, we obtain an odd function, and vice versa. We will show this in the course exercises. The proof uses the chain rule.

2.3 Scaling and shifting

In this section we list some very simple but important properties of periodic functions.

2.3.1 Amplitude scaling

Consider multiplying a periodic function $f(t)$ by an *amplitude scaling factor* $A > 0$ so that a new function $Af(t)$ is obtained. This stretches the function $f(t)$ in the y-direction. For example, $\sin(t)$ has the maximum value (or amplitude) of 1 while $2\sin(t)$ has maximum value 2. Of course we can also scale a function with a scaling factor $A < 1$, and then the function in fact shrinks in the y-direction.

2.3.2 Time shifting

Shifting the graph of a function $f(t)$ to the left by $S \geq 0$ units can be obtained by defining a new function $f(t+S)$. The new function $f(t+S)$ obtains the same values as $f(t)$ exactly S units before $f(t)$ and thus indeed shifts the graph to the *left*. Obviously the opposite effect is obtained by defining the function $f(t-S)$, which shifts the original function S units to the right.

2.3.3 Time scaling

Decreasing the period of a periodic function $f(t)$ is obtained by defining a function $f(Bt)$, where $B > 1$. We call B a *time scaling factor*. See Figure 3 for an example. The opposite effect—increasing the period—is obtained by defining $f(bt)$ for some $b < 1$ and $b > 0$.

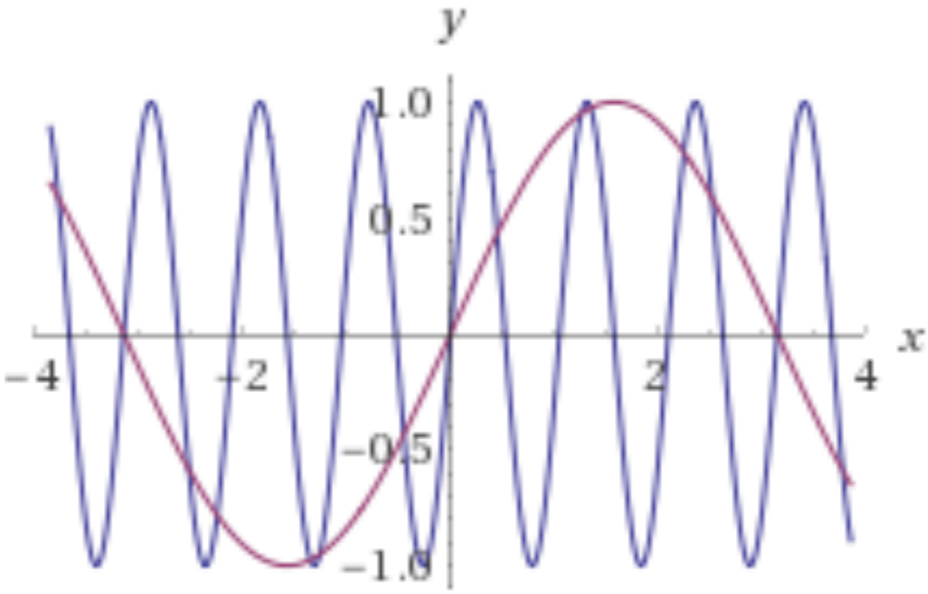

Figure 3: The figure shows $\sin(x)$ and $\sin(6x)$ plotted from -4 to 4. Here $\sin(6x)$ is the function with the blue graph, i.e., the function with a smaller fundamental period.

It is of course possible to combine the scaling and shifting operations by defining functions such as $6\sin(8t-10)$ for example.

Example 2.16. Interestingly, for example $\sin(6t-2)$ shifts $\sin(6t)$ *not* by two but only $\frac{1}{3}$ units to the right. To understand why, write $h(t) = \sin(6t)$. Now, shift $h(t)$ the amount of $\frac{1}{3}$ units to the right by defining

$$h(t-\frac{1}{3}) \;=\; \sin(6(t-\frac{1}{3})) \;=\; \sin(6t-2). \quad \square$$

3 Fourier series

Consider an infinite series

$$
\begin{aligned}
S(t) \;=\; & a_0 \cos\big(0\omega t\big) + b_0 \sin\big(0\omega t\big) + a_1 \cos\big(1\omega t\big) + b_1 \sin\big(1\omega t\big) \\
& \qquad + a_2 \cos\big(2\omega t\big) + b_2 \sin\big(2\omega t\big) + \ldots
\end{aligned}
\tag{3}
$$

where a_n, b_n and ω are real numbers. This is a very general and elegant[1] infinite series formulated in terms of trigonometric functions of *increasingly small periods*. For each positive integer n, the coefficient a_n is the amplitude and

$$\frac{2\pi}{n\omega}$$

the smallest period (or fundamental period) of the term $a_n \cos(n\omega t)$. Similarly, b_n is the amplitude and $\frac{2\pi}{n\omega}$ the smallest period of $b_n \sin(n\omega t)$. Since $\cos(0) = 1$ and $\sin(0) = 0$, we have

$$
\begin{aligned}
S(t) \;=\; & a_0 + a_1 \cos\big(1\omega t\big) + b_1 \sin\big(1\omega t\big) \\
& \qquad + a_2 \cos\big(2\omega t\big) + b_2 \sin\big(2\omega t\big) + \ldots \\
& \qquad\qquad = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big).
\end{aligned}
\tag{4}
$$

It turns out that many functions can be represented by these kinds of nicely regular infinite series of trigonometric functions, i.e., in terms of series of the form

$$a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big).$$

[1]By "very general and elegant" we mean that the series is mathematically simple and regular as opposed to arbitrarily defined, contrived and random looking.

3.1 Defining Fourier series

A **Fourier series** $S(t)$ is a series of the form

$$\frac{a_0}{2} + a_1 \cos\big(\omega t\big) + b_1 \sin\big(\omega t\big) + a_2 \cos\big(2\omega t\big) + b_2 \sin\big(2\omega t\big) \dots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big), \quad (5)$$

where a_n, b_n and ω are real numbers. Note that this series is *almost* the same as the "very general and elegant" series given by Equation 4. The difference is that here we have a coefficient $\frac{a_0}{2}$ and in Equation 4 the coefficient a_0. The reason we now begin with a term $\frac{a_0}{2}$ instead of a_0 is only to simplify some terms that occur frequently in calculations. Indeed, we could use a_0 instead of $\frac{a_0}{2}$, but choose otherwise to keep things simpler to read and write.

Let f be a periodic function and T a period of f. The **Fourier series of** f is the Fourier series

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big) \quad (6)$$

such that

1.
$$a_n = \frac{2}{T} \int_{d}^{d+T} f(t) \cos(n\omega t)\ dt \qquad (n = 0, 1, 2, \dots) \quad (7)$$

2.
$$b_n = \frac{2}{T} \int_{d}^{d+T} f(t) \sin(n\omega t)\ dt \qquad (n = 1, 2, \dots) \quad (8)$$

where

$$\omega = \frac{2\pi}{T}.$$

Concerning the notation, notice indeed that we let $\hat{f}$ denote the Fourier series of a function f (see Equation 6).

Now, some comments are in order.

1. At this point the definition may seem a bit arbitrary. Indeed, why do we define a_n and b_n in this way rather than some other way? This will be discussed later on in detail in Section 3.1.2.

2. There is no guarantee that the integrals

$$\int_{d}^{d+T} f(t)\cos(n\omega t)\ dt \text{ and } \int_{d}^{d+T} f(t)\sin(n\omega t)\ dt$$

have defined values. Whether they do or not depends essentially on f.

3. More importantly, given a value of t, it is not clear whether the series $\hat{f}(t)$ converges, and if it does, whether it converges to $f(t)$.

By definition, a series

$$S(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big)$$

converges to $v \in \mathbb{R}$ if the sequence of the partial sums

$$S_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\Big)$$

converges to v. In other words, the series $S(t)$ converges to v if

$$\lim_{N\to\infty} S_N\big(t\big) = v.$$

Intuitively, this means that we can make $S_N(t)$ arbitrarily close to v by making N sufficiently large. Then we may write $S(t) = v$.

Whether the integrals

$$\int_{d}^{d+T} f(t)\cos(n\omega t)\ dt \text{ and } \int_{d}^{d+T} f(t)\sin(n\omega t)\ dt$$

have defined values is a theoretical question that is in most cases beyond the scope of this course. The question can be mostly ignored, as in *typical real-life applications*, the integrals indeed do have defined values. The convergence of the series $\hat{f}(t)$ to $f(t)$ is a more important question, and we shall discuss it in detail later on in these notes.

3.1.1 Dirichlet conditions and Fourier's Theorem

We call a periodic function f **piecewise continuous** if f is piecewise continuous on the interval $[0, T]$ where T is the period of f.

A function is **monotonically increasing** in an interval $[a, b]$ if

$$f(t_1) \leq f(t_2)$$

for all t_1 and t_2 in $[a, b]$ such that $t_1 \leq t_2$. A function is **monotonically decreasing** in $[a, b]$ if

$$f(t_1) \geq f(t_2)$$

for all t_1 and t_2 in $[a, b]$ such that $t_1 \leq t_2$.

The notions of being **strictly increasing** and **strictly decreasing** are defined in exactly the same way as the notions of monotonically increasing and decreasing functions, with the exception of using the ordering symbols $<$ and $>$ instead of $\leq$ and $\geq$.

A function is **monotone** in an interval $[a, b]$ if it is either monotonically increasing or monotonically decreasing in $[a, b]$. This means that it is impossible for f to *both* increase and decrease in $[a, b]$, *Roughly speaking* this means that there are neither local maxima nor minima in $[a, b]$, but *strictly speaking* this is not exactly true, as we will see.

Definition 3.1. A periodic function f satisfies the **Dirichlet conditions** if

1. f is piecewice continuous and thus bounded.
2. The period interval $[d, d+T]$ of f can be partitioned into finitely many successive subintervals such that f is monotone on each of these subintervals, i.e., monotonically increasing or monotonically decreasing.

Informally, a periodic function thus satisfies the Dirichlet conditions if it is piecewise continuous and changes from being monotonically increasing to monotonically decreasing, or vice versa, only finitely many times.

Recall that a function f has a local maximum $f(c)$ at c if there exists an interval $(c-d, c+d)$ (where $d > 0$) around c such that $f(c) \geq f(r)$ for all $r \in (c-d, c+d)$. *For functions typical in applications, the second condition of Definition 3.1 above can be replaced by*

2.' f has finitely many local maxima and minima in $[d, d+T]$.

However, for example the constant function $f(t) = 0$ satisfies 2 but not 2'.

It is now possible to prove the Fourier's theorem:

Theorem 3.2. *If a T-periodic function f satisfies the Dirichlet conditions, then the Fourier series $\hat{f}(t)$ defined by Eqn 6 exists (i.e., the integrals have defined values), and furthermore, the series $\hat{f}(t)$ converges such that we have*

$$\hat{f}(t) = \begin{cases} f(t) & \text{if } f \text{ is continous at } t \\ \dfrac{f(t^+) + f(t^-)}{2} & \text{if } f \text{ is discontinous at } t \ . \end{cases}$$

This is a very powerful theorem, as typical functions relevant for applications indeed satisfy the Dirichlet conditions. A full proof of Fourier's theorem is beyond the scope of this course.

3.1.2 Coefficients of Fourier series

We now justify that the definition of Fourier series given by Equation 6 together with Equations 7 and 8. The point is to show that if there exists any real number coefficients $a_0, a_1, a_2 \ldots$ and $b_1, b_2 \ldots$ such that the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big)$$

converges to $f(t)$, then the coefficients *must* be defined exactly as given by Equations 7 and 8. That is, there is no alternative here, so the definition given by Equations 7 and 8 is forced upon us.

To this end, we need the following integrals. The integrals are proved in Section 3.1.3.

Orthogonality integrals:

$$\int\limits_{d}^{d+T} \cos(n\omega t)\cos(k\omega t)dt = \begin{cases} 0 & \text{for } n \neq k,\ n > 0,\ k \geq 0. \\ \frac{T}{2} & \text{for } n = k > 0 \end{cases} \tag{9}$$

$$\int\limits_{d}^{d+T} \sin(n\omega t)\sin(k\omega t)dt = \begin{cases} 0 & \text{for } n \neq k \\ \frac{T}{2} & \text{for } n = k > 0 \end{cases} \tag{10}$$

$$\int\limits_{d}^{d+T} \sin(n\omega t)\cos(k\omega t)dt = 0 \tag{11}$$

where $T = \frac{2\pi}{\omega}$.

We also need the following integrals that will, likewise, be proved in Section 3.1.3. (The integrals are easy to evaluate by a direct calculation, but we shall prove the below identities differently.)

$$\int\limits_{d}^{d+T} \cos(n\omega t)dt = \begin{cases} 0 & \text{when } \ n \neq 0 \\ T & \text{when } \ n = 0 \end{cases} \tag{12}$$

$$\int\limits_{d}^{d+T} \sin(n\omega t)dt = \ 0 \tag{13}$$

where $T = \frac{2\pi}{\omega}$.

Now suppose that f satisfies the Dirichlet conditions.[2] Let T be the period of f. Suppose that for all

$$t \in [d, d+T],$$

with the possible exception of a finite number of values of t, we have

$$f(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big), \tag{14}$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are some real numbers. We next show that under the assumption that Equation 14 indeed holds, we *necessarily* have

1.
$$a_n \;=\; \int\limits_{d}^{d+T} f(t)\cos(n\omega t)\ dt \qquad (n = 0, 1, 2, \ldots)$$

2.
$$b_n \;=\; \int\limits_{d}^{d+T} f(t)\sin(n\omega t)\ dt \qquad (n = 1, 2, \ldots).$$

In other words, when $f(t)$ converges to the series given on the right hand side of Equation 14, *the constants a_n and b_n must be chosen exactly as dictated by Equations 7 and 8, and thus the definition of $\hat{f}$ (given by Equations 6, 7 and 8) could not be done in any other way.* There is no alternative here for us, as there are no other similarly converging Fourier series with alternative definitions for a_n and b_n. This justifies the definition of a_n and b_n that might otherwise seem somewhat arbitrary.

Case 1: showing that $\mathbf{a_n} = \frac{\mathbf{2}}{\mathbf{T}} \int\limits_{\mathbf{d}}^{\mathbf{d+T}} \mathbf{f(t)}\cos(\mathbf{n}\omega\mathbf{t})\ \mathbf{dt} \qquad (\mathbf{n = 0, 1, 2, \ldots})$

Let k be a nonnegative integer. Multiply Equation 14 by $\cos(k\omega t)$ and integrate from d to $d+T$. Thus we obtain the following equation:

[2]We note that there are other, less restrictive constraints that could be assumed here instead of the Dirichlet conditions, and our argument would still go through.

$$\int_{d}^{d+T} f(t)\cos(k\omega t)\,dt$$

$$= \int_{d}^{d+T} \frac{a_0}{2}\cos(k\omega t)\,dt \;+\; \sum_{n=1}^{\infty}\Bigg(\; a_n \int_{d}^{d+T} \cos\big(n\omega t\big)\cos\big(k\omega t\big)\,dt$$

$$+\; b_n \int_{d}^{d+T} \sin\big(n\omega t\big)\cos\big(k\omega t\big)\,dt \;\Bigg).$$

We note that there are infinite sums $\sum\limits_{n=1}^{\infty} g_n(t)$ such that $\int_u^v(\sum\limits_{n=1}^{\infty} g_n(t))dt$ cannot be replaced by $\sum\limits_{n=1}^{\infty}(\int_u^v g_n(t)dt)$ in the way we did here. However, it can be shown that here this exchange is fine. A fully rigorous justification of the exchange is beyond the level of this course.

Using the orthogonality integrals (Equations 9, 10, 11), and the integral of Equation 12, we see that

$$\int_d^{d+T} f(t)\cos(k\omega t)\,dt$$

$$= \underbrace{\int_d^{d+T} \frac{a_0}{2}\cos(k\omega t)\,dt}_{\begin{cases} 0 & \text{if } k \neq 0 \\ a_0\,\frac{T}{2} & \text{if } k = 0 \end{cases}} \;+\; \sum_{n=1}^{\infty}\Bigg(\; a_n \underbrace{\int_d^{d+T} \cos\big(n\omega t\big)\cos\big(k\omega t\big)\,dt}_{=\begin{cases} 0 & \text{if } k \neq n \\ \frac{T}{2} & \text{if } k = n \\ \text{(note: } n \neq 0 \text{ here)} \end{cases}}$$

$$+\; b_n \underbrace{\int_d^{d+T} \sin\big(n\omega t\big)\cos\big(k\omega t\big)\,dt}_{=0} \;\Bigg).$$

Therefore

$$\int_d^{d+T} f(t)\cos(k\omega t)\,dt \;=\; a_k\,\frac{T}{2},$$

whence

$$a_k \;=\; \frac{2}{T}\int_d^{d+T} f(t)\cos(k\omega t)\,dt.$$

Thus

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos(n\omega t)\,dt.$$

Case 2: showing that $\mathbf{b_n = \frac{2}{T} \int_d^{d+T} f(t)\sin(n\omega t)\ dt \qquad (n = 1, 2, \dots)}$

Let $k \geq 1$. This time we multiply Equation 14 by $\sin(k\omega t)$ and integrate from d to $d+T$, obtaining the following equation:

$$\int_{d}^{d+T} f(t)\sin(k\omega t)\,dt$$

$$= \int_{d}^{d+T} \frac{a_0}{2}\sin(k\omega t)\,dt \;+\; \sum_{n=1}^{\infty}\Big(\; a_n \int_{d}^{d+T} \cos\big(n\omega t\big)\sin\big(k\omega t\big)\,dt$$

$$+\;\; b_n \int_{d}^{d+T} \sin\big(n\omega t\big)\sin\big(k\omega t\big)\,dt \;\Big).$$

Using the orthogonality integrals (Equations 9, 10, 11) and the integral from Equation 13, we observe that

$$\int_{d}^{d+T} f(t)\sin(k\omega t)\,dt$$

$$= \underbrace{\int_{d}^{d+T} \frac{a_0}{2}\sin(k\omega t)\,dt}_{=0} \;+\; \sum_{n=1}^{\infty}\Big(\; a_n \underbrace{\int_{d}^{d+T} \cos\big(n\omega t\big)\sin\big(k\omega t\big)\,dt}_{=0}$$

$$+\;\; b_n \underbrace{\int_{d}^{d+T} \sin\big(n\omega t\big)\sin\big(k\omega t\big)\,dt}_{\begin{cases} 0 & \text{if } n \neq k \\ \frac{T}{2} & \text{if } n = k > 0 \end{cases}} \;\Big).$$

Therefore

$$\int_{d}^{d+T} f(t)\sin(k\omega t)\,dt \;=\; b_k\,\frac{T}{2},$$

whence

$$b_k \;=\; \frac{2}{T}\int_{d}^{d+T} f(t)\sin(k\omega t)\,dt.$$

Thus

$$b_n = \frac{2}{T} \int_{d}^{d+T} f(t) \sin(n\omega t)\, dt.$$

3.1.3 Orthogonality integrals

In this section we prove the orthogonality integrals from Equations 9, 10 and 11. Before that, we prove Equations 12 and 13.[3]

We first show that

$$\int_{d}^{d+T} \sin(n\omega t) dt = 0. \tag{15}$$

where $n = 0, 1, 2, \ldots$ and $T = \frac{2\pi}{\omega}$.

If $n = 0$, this is clear. Thus we assume that $n \neq 0$. We note that $\sin(n\omega t)$ has period

$$\frac{2\pi}{n\omega} = \frac{2\pi}{n\frac{2\pi}{T}} = \frac{T}{n},$$

and therefore also T is a period of $\sin(n\omega T)$, being a multiple of $\frac{T}{n}$. Thus

$$\int_{d}^{d+T} \sin(n\omega t) dt$$

is an integral over a full period of the function $\sin(n\omega t)$. Thus clearly

$$\int_{d}^{d+T} \sin(n\omega t) dt = 0.$$

[3] In this section we will not always attempt to use the fastest methods of proving the integrals we deal with, but instead use some elucidating, longer approaches that are, nevertheless, reasonably short. We also occasionally *repeat* some interesting points.

To see why this is clear, note that sine is an odd function, and integrating odd functions over a *any period* gives zero by Equation 1 and Theorem 2.11. Indeed, recall that Equation 1 stated that integrating an odd function over a symmetric interval $[-c, c]$ gives zero, while Theorem 2.11 stated that when integrating a periodic function over a period, it does not matter where we place the interval of integration. While the period interval $[d, d+T]$ may not be symmetric (i.e., of the form $[-c, c]$), Theorem 2.11 indeed shows that we can shift the integration interval.

We then show that

$$\int_{d}^{d+T} \cos(n\omega t)dt = \begin{cases} 0 & \text{when} \quad n \neq 0 \\ T & \text{when} \quad n = 0 \end{cases} \tag{16}$$

where $n = 0, 1, 2, \ldots$ and $T = \frac{2\pi}{\omega}$.

The case with $n = 0$ is clear, so we assume that $n \neq 0$. We note that $\cos(n\omega t)$ has period

$$\frac{2\pi}{n\omega} = \frac{2\pi}{n\frac{2\pi}{T}} = \frac{T}{n}.$$

Thus also T is a period of $\cos(n\omega T)$, being a multiple of $\frac{T}{n}$. Therefore

$$\int_{d}^{d+T} \cos(n\omega t)dt$$

is an integral over a full period of the function $\cos(n\omega t)$ based on the cosine function. Thus clearly

$$\int_{d}^{d+T} \cos(n\omega t)dt = 0.$$

(To see why this is clear, note that $\sin(n\omega t)$ and $\cos(n\omega t)$ differ only by a time shifting factor or phase shift (see Section 2.3.2), so their integrals over a period are the same, zero.)

First orthogonality integral

We now calculate the orthogonality integrals given in Equations 9, 10 and 11. We first show that

$$\int_d^{d+T} \cos\big(n\omega t\big)\cos\big(k\omega t\big)\,dt \;=\; \begin{cases} 0 & \text{if } n \neq k,\ n > 0, k \geq 0 \\ \frac{T}{2} & \text{if } n = k > 0. \end{cases} \tag{17}$$

Assume that $n \neq k,\ n > 0,\ k \geq 0$. Using the well known trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b)),$$

we see that now

$$\begin{aligned} &\int_d^{d+T} \cos\big(n\omega t\big)\cos\big(k\omega t\big)\,dt \\ =\;& \frac{1}{2}\Big(\int_d^{d+T} \cos\big(n\omega t + k\omega t\big)\,dt + \int_d^{d+T} \cos\big(n\omega t - k\omega t\big)\,dt\Big) \\ =\;& \frac{1}{2}\Big(\int_d^{d+T} \cos\big((n+k)\omega t\big)\,dt + \int_d^{d+T} \cos\big((n-k)\omega t\big)\,dt\Big). \end{aligned} \tag{18}$$

Now recall that for any positive constant c, the function $\cos(ct)$ has a period $\frac{2\pi}{c}$, so $\cos\big((n+k)\omega t\big)$ has a period

$$\frac{2\pi}{(n+k)\omega} = \frac{2\pi}{(n+k)\frac{2\pi}{T}} = \frac{T}{n+k}.$$

Therefore also

$$(n+k)\frac{T}{n+k} \;=\; T$$

is a period of $\cos\big((n+k)\omega t\big)$. We have thus deduced that T is a period of $\cos\big((n+k)\omega t\big)$, and we can—with the same argument—show that T is also a period of $\cos\big((n-k)\omega t\big) = \cos\big(-(n-k)\omega t\big) = \cos\big((k-n)\omega t\big)$. Therefore the integrals

$$\int_{d}^{d+T} \cos\big((n+k)\omega t\big)\,dt \text{ and } \int_{d}^{d+T} \cos\big((n-k)\omega t\big)$$

from Line 18 are integrals over a full period. Since these functions based on the cosine function, the integrals must be equal to zero. That is, we have

$$\int_{d}^{d+T} \cos\big((n+k)\omega t\big)\,dt \;= 0 \;\text{ when } n \neq k,\; n > 0,\; k \geq 0. \tag{19}$$

$$\text{and } \int_{d}^{d+T} \cos\big((n-k)\omega t\big) \;= 0 \;\text{ when } n \neq k,\; n > 0,\; k \geq 0. \tag{20}$$

Thus we have shown that the integral of Equation 17 is indeed zero when $n \neq k,\; n > 0, k \geq 0$.

We then prove the integral in Equation 17 in the case where $n = k > 0$. Again using the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b)),$$

we have

$$\begin{aligned}
\int_{d}^{d+T} \cos\big(n\omega t\big)\cos\big(k\omega t\big)\,dt = &\quad \int_{d}^{d+T} \cos^2\big(n\omega t\big)\,dt \qquad (\text{recall } n = k)\\
= &\quad \frac{1}{2}\Big(\int_{d}^{d+T} \cos\big(n\omega t + n\omega t\big)\,dt + \int_{d}^{d+T} \cos\big(n\omega t - n\omega t\big)\,dt\;\Big)\\
= &\quad \frac{1}{2}\Big(\int_{d}^{d+T} \cos\big(2n\omega t\big)\,dt + \int_{d}^{d+T} 1\,dt\;\Big)\\
= &\quad \frac{1}{2}\Big(\int_{d}^{d+T} \cos\big(2n\omega t\big)\,dt\Big) + \frac{T}{2}.
\end{aligned}$$

Now, $\int\limits_{d}^{d+T} \cos\big(2n\omega t\big)\, dt \ = 0$, as $\cos\big(2n\omega t\big)$ has period

$$\frac{2\pi}{2n\omega} = \frac{2\pi}{2n\frac{2\pi}{T}} = \frac{T}{2n}$$

and therefore also T is a period of $\cos\big(2n\omega t\big)$. We have thus now shown that

$$\int\limits_{d}^{d+T} \cos^2\big(n\omega t\big)\, dt \ = \ \frac{T}{2}$$

and thereby proved that Equation 17 indeed holds.

Second orthogonality integral

We then prove the second orthogonality integral given by Equation 10, that is, we will prove that

$$\int_{d}^{d+T} \sin(n\omega t)\sin(k\omega t)dt = \begin{cases} 0 & \text{for } n \neq k \\ \frac{T}{2} & \text{for } n = k > 0\ . \end{cases} \tag{21}$$

First, if n or k is zero, we clearly have $\int_d^{d+T} \sin(n\omega t)\sin(k\omega t)dt = 0$. Thus we only have to deal with the cases where $n > 0$ and $k > 0$.

We first assume that $n \neq k$. By the trigonometric identity

$$\sin(a)\sin(b) = \frac{1}{2}(-\cos(a+b) + \cos(a-b)),$$

we have

$$\begin{aligned} & \int_{d}^{d+T} \sin\big(n\omega t\big)\sin\big(k\omega t\big)\,dt \\ = \quad & \frac{1}{2}\Big(-\int_{d}^{d+T} \cos\big(n\omega t + k\omega t\big)\,dt + \int_{d}^{d+T} \cos\big(n\omega t - k\omega t\big)\,dt\ \Big) \\ = \quad & \frac{1}{2}\Big(-\int_{d}^{d+T} \cos\big((n+k)\omega t\big)\,dt + \int_{d}^{d+T} \cos\big((n-k)\omega t\big)\,dt\ \Big). \end{aligned} \tag{22}$$

By Equations 19 and 20, we see that both of these integrals are zero. Thus we have now covered the first case of Equation 21. To cover the second case, we assume that $n = k > 0$.

Once more using the trigonometric identity

$$\sin(a)\sin(b) = \frac{1}{2}(-\cos(a+b) + \cos(a-b)),$$

we have

$$\begin{aligned}
&\int_{d}^{d+T} \sin^2\big(n\omega t\big)\,dt \\
&= \frac{1}{2}\Big(-\int_{d}^{d+T} \cos\big(n\omega t + n\omega t\big)\,dt + \int_{d}^{d+T} \cos\big(n\omega t - n\omega t\big)\,dt\ \Big) \\
&= \frac{1}{2}\Big(-\int_{d}^{d+T} \cos\big(2n\omega t\big)\,dt + \int_{d}^{d+T} 1\,dt\ \Big) \\
&= -\frac{1}{2}\int_{d}^{d+T} \cos\big(2n\omega t\big)\,dt + \frac{T}{2}.
\end{aligned}$$

We know that the integral $\int_{d}^{d+T} \cos\big(2n\omega t\big)\,dt$ is zero (as $\cos\big(2n\omega t\big)$ has period $\frac{2\pi}{2n\omega} = \frac{2\pi}{2n\frac{2\pi}{T}} = \frac{T}{2n}$ and thus also T is a period of $\cos\big(2n\omega t\big)$). We have thereby now shown that

$$\int_{d}^{d+T} \sin^2\big(n\omega t\big)\,dt \ = \ \frac{T}{2}$$

whence Equation 21 indeed holds.

Third orthogonality integral

Here we prove that

$$\int_{d}^{d+T} \sin(n\omega t)\cos(k\omega t)dt = 0.$$

If n is zero, this is clear. If k is zero (but n is not), this is an integral of $\sin(n\omega t)$ over T, which is a period of $\sin(n\omega t)$, as $\frac{2\pi}{n\omega} = \frac{2\pi}{n\frac{2\pi}{T}} = \frac{T}{n}$ is a period. Thus we then assume that neither n nor k is zero.

Now, arguing once again the same way as we have done many times above, we see that T is a period of both $\sin(n\omega t)$ and $\cos(k\omega t)$. (Indeed, $\frac{2\pi}{n\omega} = \frac{2\pi}{n\frac{2\pi}{T}} = \frac{T}{n}$ is a period of $\sin(n\omega t)$ and $\frac{2\pi}{k\omega} = \frac{2\pi}{k\frac{2\pi}{T}} = \frac{T}{k}$ a period of $\cos(k\omega t)$, and thereby T must be a period of both of these functions, being a multiple of both $\frac{T}{n}$ and $\frac{T}{k}$.)

Since T is a period of both of these functions, it is a period of their product $\sin(n\omega t)\cos(k\omega t)$. As

$$\sin(n\omega t)\cos(k\omega t)$$

is a product of an odd and an even function, it is an odd function, and thus

$$\int_{d}^{d+T} \sin(n\omega t)\cos(k\omega t)dt$$

is an integral of an odd function over one period T. Thus we must have

$$\int_{d}^{d+T} \sin(n\omega t)\cos(k\omega t)dt \ = \ 0.$$

We have now covered all the orthogonality integrals.

3.2 Fourier series, first examples

Example 3.3. Let f be the function defined such that

1. $$f(t) = \begin{cases} 0 & \text{if } -\pi \leq t < 0 \\ 1 & \text{if } 0 \leq t < \pi. \end{cases}$$

2. $$f(t + 2\pi) = f(t) \text{ for all } t.$$

Let us find the Fourier series $\hat{f}$ of this function.

By Eqn 6, the Fourier series is

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big)$$

where

$$a_n = \frac{2}{T} \int_d^{d+T} f(t) \cos(n\omega t)\, dt \qquad (n = 0, 1, 2, \dots)$$

and

$$b_n = \frac{2}{T} \int_d^{d+T} f(t) \sin(n\omega t)\, dt \qquad (n = 1, 2, \dots).$$

The function f has period 2π, so we put $T = 2\pi$ and recall that $\omega = \dfrac{2\pi}{T}$, so $\omega = 1$. The value of d can be chosen freely; we choose $d = -\pi$.

Therefore

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \;=\; \frac{1}{\pi}\Big(\int_{-\pi}^{0} 0\cdot\cos(nt)\,dt \;+\; \int_{0}^{\pi} 1\cdot\cos(nt)\,dt\;\Big)$$

$$= \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos(nt)\,dt \;=\begin{cases} \dfrac{1}{\pi}\Big/_{-\pi}^{0}\dfrac{1}{n}\sin(nt) = 0-0=0 & \text{for } n\neq 0 \\ \dfrac{1}{\pi}\Big/_{-\pi}^{0} t \quad = \;1 & \text{for } n=0 \end{cases}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\Big(\int_{-\pi}^{0} 0\cdot\sin(nt)\,dt \;+\; \int_{0}^{\pi} 1\cdot\sin(nt)dt\Big)$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt$$

$$= \;-\frac{1}{\pi}\Big/_{0}^{\pi}\frac{1}{n}\cos(nt)$$

$$= -\,\frac{1}{n\pi}\Big(\;(-1)^n-1\;\Big)$$

$$= \begin{cases} 0 & \text{for even } n \\ \dfrac{2}{n\pi} & \text{for odd } n\,. \end{cases}$$

Therefore the Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \big(0 \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin\big(nt\big) \qquad\qquad \text{(recall that } \omega = 1\text{)}$$

$$= \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{2}{n\pi} \sin\big(nt\big)$$

$$= \frac{1}{2} + \frac{2}{\pi}\Big(\frac{\sin\big(t\big)}{1} + \frac{\sin\big(3t\big)}{3} + \frac{\sin\big(5t\big)}{5} + \ \ldots \ \Big).$$

This concludes Example 3.3. $\square$

Example 3.4. It would be desirable that the Fourier series of the cosine function was the cosine function itself. In this example we verify that this is indeed the case. More exactly, we will show that the Fourier series of $\cos(m\omega t)$ is $\cos(m\omega t)$. Here m denotes a positive integer.

To obtain the Fourier series, let us first find a_0. We get

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T \cos\big(m\omega t\big)\,dt \\
&= \begin{cases} \dfrac{2}{T}\Big/_0^T \dfrac{1}{m\omega}\sin\big(m\omega t\big) & \text{if } m \neq 0 \\[2ex] \dfrac{2}{T}\Big/_0^T t & \text{if } m = 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } m \neq 0 \\ 2 & \text{if } m = 0. \end{cases}
\end{aligned}
$$

However, we defined that m is a positive integer in our example, so we conclude that $a_0 = 0$.

Now recall that $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$. Using this, we see that the coefficients a_n and b_n for $n \geq 1$ are

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T \cos\big(m\omega t\big)\cos\big(n\omega t\big)\,dt \\
&= \frac{1}{T}\int_0^T \Big(\cos\big((m+n)\omega t\big) + \cos\big((m-n)\omega t\big)\Big)\,dt \\
&= \begin{cases} \dfrac{1}{T}\Big/_0^T \left(\dfrac{1}{(m+n)\omega}\sin\big((m+n)\omega t\big) + \dfrac{1}{(m-n)\omega}\sin\big((m-n)\omega t\big)\right) & \text{when } m \neq n \\[2ex] \dfrac{1}{T}\Big/_0^T \left(\dfrac{1}{2m\omega}\sin\big(2m\omega t\big) + t\right) & \text{when } m = n \end{cases} \\
&= \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n. \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\cos\big(m\omega t\big)\sin\big(n\omega t\big)}_{\text{an odd function}}\,dt \\
&= 0.
\end{aligned}
$$

Thus the Fourier series of $\cos(m\omega t)$ is

$$
\begin{aligned}
\hat{f}\big(t\big) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos\big(n\omega t\big) + b_n\sin\big(n\omega t\big)\Big) \\
&= \cos\big(m\omega t\big),
\end{aligned}
$$

as desired. $\square$

Example 3.5. This example is similar to the previous one, albeit even easier. We will show that the Fourier series of a constant function $f(t) = c$ is the constant $c \in \mathbb{R}$ itself. This is a very useful result, as it can be directly used for obtaining a Fourier series $\hat{g}(t)$ of a function $g(t) = h(t)+c$ from an already known Fourier series $\hat{h}(t)$ of $h(t)$. This means obtaining the Fourier series of a function obtained by vertically shifting some function whose Fourier series we already know. We shall discuss how $\hat{g}(t)$ is obtained from $\hat{h}(t)$ in Section 3.4, Example 3.7.

Now let us verify that the Fourier series of $f(t) = c$ is c. The constant function f has every possible period, so we choose to work with the period $T = 2\pi$. Thus $\omega = \frac{2\pi}{T} = 1$. We have

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^{2\pi} c\ dt \ = \ \frac{2}{2\pi}\big(2\pi c\big) = 2c, \\
a_n &= \frac{2}{T}\int_0^{2\pi} c\cos(nt)dt \ = \ 0 \text{ for } n \geq 1, \\
b_n &= \frac{2}{T}\int_0^{2\pi} c\sin(nt)dt \ = \ 0 \text{ for } n \geq 1.
\end{aligned}
$$

Therefore the Fourier series is is

$$\begin{aligned}\hat{f}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big) \Big) \\ &= \frac{2c}{2} = c,\end{aligned}$$

as expected. $\square$

3.3 Exponential form of Fourier series

In this section we discuss the exponential form of Fourier series. It is more compact than the trigonometric form and it is therefore sometimes advantageous to use it. Especially calculations can often become easier. However, we must make use of the complex numbers when dealing with this representation.

Recall that

$$e^{jx} \;=\; \cos(x) \;+ j\sin(x), \tag{23}$$

where j stands for the standard complex square root of -1. Thus, recalling that

$$\cos(-x) = \cos(x) \text{ and } \sin(-x) = -\sin(x),$$

we have

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{24}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}. \tag{25}$$

Therefore we can write

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\Big(\frac{e^{jn\omega t} + e^{-jn\omega t}}{2}\Big) + b_n\Big(\frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}\Big)\Big)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(\frac{a_n}{2}e^{jn\omega t} + \frac{a_n}{2}e^{-jn\omega t} + \frac{b_n}{2j}e^{jn\omega t} - \frac{b_n}{2j}e^{-jn\omega t} \Big)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(\frac{a_n}{2}e^{jn\omega t} + \frac{a_n}{2}e^{-jn\omega t} - j\frac{b_n}{2}e^{jn\omega t} + j\frac{b_n}{2}e^{-jn\omega t} \Big) \text{ (recall } \frac{1}{j} = -j)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big((\frac{a_n - jb_n}{2})e^{jn\omega t} + (\frac{a_n + jb_n}{2})e^{-jn\omega t} \Big)$$

$$= c_0 + \sum_{n=1}^{\infty}\big(c_n e^{jn\omega t} + c_{-n}e^{-jn\omega t} \big) \quad \text{(Here we simply } \textit{define} \text{ new symbols } c_0, c_n, c_{-n}.)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

where

$$\begin{aligned} c_0 &= \frac{a_0}{2} \\ c_n &= \frac{a_n - jb_n}{2} && \text{for } n = 1, 2, 3 \ldots \\ c_n &= \frac{a_{-n} + jb_{-n}}{2} && \text{for } n = -1, -2, -3 \ldots \end{aligned}$$

We have thus shown that

$$\hat{f}(t) \;=\; \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}, \tag{26}$$

or alternatively,

$$\hat{f}(t) \;=\; c_0 + \sum_{n=1}^{\infty} \Big(\, c_n e^{jn\omega t} + c_{-n} e^{-jn\omega t} \, \Big). \tag{27}$$

We also note that, when $n \geq 1$, we have

$$\int_d^{d+T} f(t) e^{-jn\omega t}\, dt = \int_d^{d+T} f(t) \Big(\; \cos(-n\omega t) \; + j \sin(-n\omega t) \Big)\, dt$$

$$= \int_d^{d+T} f(t) \cos(n\omega t)\, dt \; - \; j \int_d^{d+T} f(t) \sin(n\omega t)\, dt$$

$$= \frac{T}{2}(a_n - \; jb_n) \qquad \text{(by Eqns 7 and 8)}$$

$$= Tc_n \qquad \text{(since } n \geq 1\text{; see the definition of } c_n \text{ on the previous page.)}$$

Furthermore, when $n \leq -1$, then, writing $k = -n$, we have

$$\int_{d}^{d+T} f(t)e^{-jn\omega t}\,dt = \int_{d}^{d+T} f(t)\Big(\ \cos(-n\omega t)\ + j\sin(-n\omega t)\Big)\,dt$$

$$= \int_{d}^{d+T} f(t)\cos(k\omega t)\,dt\ +\ j\int_{d}^{d+T} f(t)\sin(k\omega t)\,dt$$

$$= \frac{T}{2}(a_k +\ jb_k) \quad \text{(by Eqns 7 and 8 and since } k \geq 1)$$

$$= \frac{T}{2}(a_{-n} +\ jb_{-n}) \quad \text{(since } -n = k \geq 1)$$

$$= Tc_n \quad \text{(since } n \leq 1).$$

Finally, when $n = 0$, then

$$\int_{d}^{d+T} f(t)e^{-jn\omega t}\,dt = \int_{d}^{d+T} f(t)e^{j\cdot 0\cdot \omega t}\,dt$$

$$= \int_{d}^{d+T} f(t)\,dt$$

$$= \frac{T}{2}a_0$$

$$= Tc_0\ .$$

Therefore we have shown that for all n (positive, negative, zero), we have

$$\int_{d}^{d+T} f(t)e^{-jn\omega t} \;=\; Tc_n$$

and thus

$$c_n \;=\; \frac{1}{T}\int_{d}^{d+T} f(t)e^{-jn\omega t}.$$

Combining this with Equation 26, we have

$$\boxed{\begin{aligned} \hat{f}(t) \;&=\; \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} && (28) \\ c_n \;&=\; \frac{1}{T}\int_{d}^{d+T} f(t)e^{-jn\omega t}. && (29) \end{aligned}}$$

Alternatively, using Equation 27, we have

$$\boxed{\begin{aligned} \hat{f}(t) \;&=\; c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\omega t} + c_{-n}e^{-jn\omega t} \right) && (30) \\ c_n \;&=\; \frac{1}{T}\int_{d}^{d+T} f(t)e^{-jn\omega t}. && (31) \end{aligned}}$$

We have thus identified the **exponential forms** of Fourier series.

For converting Fourier series from the exponential form to the trigonometric form, it is useful to remember the above deduced identities:

$$c_0 = \frac{a_0}{2} \tag{32}$$

$$c_n = \frac{a_n - jb_n}{2} \quad \text{for } n = 1, 2, 3 \dots \tag{33}$$

$$c_n = \frac{a_{-n} + jb_{-n}}{2} \quad \text{for } n = -1, -2, -3 \dots \tag{34}$$

Notice that c_{-n} is the complex conjugate of c_n.

3.4 Linearity of Fourier series

In this section we prove that if a function f can be represented as a linear combination of functions g and h, then the Fourier series $\hat{f}$ is representable as a linear combination of $\hat{g}$ and $\hat{h}$. This if formally captured by the following theorem.

Theorem 3.6. *Let a and b be constants, and let f, g and h be T-periodic functions. Suppose that*

$$f(t) = ag(t) + bh(t).$$

Then

$$\hat{f}(t) = a\hat{g}(t) + b\hat{h}(t).$$

Proof. We write

$$1. \quad \hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$2. \quad \hat{g}(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega t}$$

$$3. \quad \hat{h}(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega t}.$$

By Equation 29, we know that

$$c_n = \frac{1}{T}\int\limits_{d}^{d+T} f(t)e^{-jn\omega t}dt$$

$$\alpha_n = \frac{1}{T}\int\limits_{d}^{d+T} g(t)e^{-jn\omega t}dt$$

$$\beta_n = \frac{1}{T}\int\limits_{d}^{d+T} h(t)e^{-jn\omega t}dt.$$

Therefore

$$c_n = \frac{1}{T}\int_{d}^{d+T} f(t)e^{-jn\omega t}\,dt$$

$$= \frac{1}{T}\int_{d}^{d+T} \Big(ag(t) + bh(t)\Big)e^{-jn\omega t}\,dt$$

$$= a\,\frac{1}{T}\int_{d}^{d+T} g\big(t\big)e^{-jn\omega t}\,dt + b\,\frac{1}{T}\int_{d}^{d+T} h\big(t\big)e^{-jn\omega t}\,dt$$

$$= a\alpha_n + b\beta_n.$$

Thus

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty} (a\alpha_n + b\beta_n)e^{jn\omega t}$$

$$= a\sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega t} + b\sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega t}$$

$$= a\hat{g}\big(t\big) + b\hat{h}\big(t\big).$$

This concludes the proof. $\square$

Example 3.7. Theorem 3.6 can be very useful for obtaining Fourier series of functions from known Fourier series of other functions. We now present an example of this.

Let us find the Fourier series of

1.
$$g(t) \;=\; \begin{cases} -\frac{1}{2} & \text{if } \; -\pi \leq t < 0 \\ \frac{1}{2} & \text{if } \; 0 \leq t < \pi. \end{cases}$$

2.
$$g(t + 2\pi) = g(t) \text{ for all } t.$$

Now, this function g is similar to the block wave f whose Fourier series we found in Example 3.3. We have $g(t) = f(t) - \frac{1}{2}$ for all t, so g obtained by shifting f downwards by $\frac{1}{2}$ units. The Fourier series of f is

$$\hat{f}(t) \;=\; \frac{1}{2} + \frac{2}{\pi}\Big(\frac{\sin{(t)}}{1} + \frac{\sin{(3t)}}{3} + \frac{\sin{(5t)}}{5} + \;\ldots\; \Big)$$

and the Fourier series of $\frac{-1}{2}$ is $\frac{-1}{2}$ (by example 3.5). Thus, by Theorem 3.6, the Fourier series of $g(t) = f(t) - \frac{1}{2}$ is $\hat{g}(t) = \hat{f}(t) - \frac{1}{2}$. That is, we have

$$\hat{g}(t) \;=\; \frac{2}{\pi}\Big(\frac{\sin{(t)}}{1} + \frac{\sin{(3t)}}{3} + \frac{\sin{(5t)}}{5} + \;\ldots\; \Big).$$

Thus we managed to find the Fourier series $\hat{g}(t)$ very easily by simply using the linearity property of Fourier series given by Theorem 3.6. □

Example 3.8. Let us find the Fourier series of

1.
$$h(t) \;=\; \begin{cases} -\frac{1}{2} & \text{if } \; -\pi \leq t < 0 \\ 0 & \text{if } \; 0 \leq t < \pi. \end{cases}$$

2.
$$h(t + 2\pi) = h(t) \text{ for all } t.$$

Again we start from the Fourier series of the block wave f of Example 3.3 i.e., the series

$$\hat{f}(t) \;=\; \frac{1}{2} + \frac{2}{\pi}\Big(\frac{\sin\big(t\big)}{1} + \frac{\sin\big(3t\big)}{3} + \frac{\sin\big(5t\big)}{5} + \;\dots\; \Big).$$

This time we have $h(t) = \frac{1}{2}f(t) - \frac{1}{2}$ for all t. Thus we obtain h from f by first scaling f vertically and then shifting the result downwards by $\frac{1}{2}$ units. By linearity, we have $\hat{h}(t) = \frac{1}{2}\hat{f}(t) - \frac{1}{2}$. Therefore

$$\hat{h}(t) \;=\; -\frac{1}{4} + \frac{1}{\pi}\Big(\frac{\sin\big(t\big)}{1} + \frac{\sin\big(3t\big)}{3} + \frac{\sin\big(5t\big)}{5} + \;\dots\; \Big).$$

Again finding the Fourier series was very easy using the linearity property.
□

Linearity is indeed a very useful property, especially when finding Fourier series of functions that are vertically shifted (as in Example 3.7) or scaled (as in Example 3.8).

Example 3.9. Let us verify that the Fourier series of $5 + \frac{1}{3}\cos(4t)$ is the function $5 + \frac{1}{3}\cos(4t)$ itself. In Example 3.4, we found out that the Fourier series of $\cos(m\omega t)$ is $\cos(m\omega t)$, so the Fourier series of $\cos(4t)$ is $\cos(4t)$. By linearity, the Fourier series of $\frac{1}{3}\cos(4t)$ is therefore $\frac{1}{3}\cos(4t)$ itself. In Example 3.5, we indeed verified that the Fourier series of a constant $c \in \mathbb{R}$ is the constant c itself, so the Fourier series of 5 is 5. Therefore, by linearity, the Fourier series of $5 + \frac{1}{3}\cos(4t)$ is $5 + \frac{1}{3}\cos(4t)$.

3.4.1 Fourier series of time shifted and time scaled functions

Above we saw that by linearity, the amplitude scaled function $sf(t)$ (where $s \in \mathbb{R}$) has the Fourier series $s\hat{f}(t)$ and the vertically shifted function $f(t)+s$ the Fourier series $\hat{f}(t)+s$. We now investigate Fourier series of *time scaled* functions $f(st)$ (where $s > 0$ is a constant in $\mathbb{R}$) as well as *time shifted* functions $f(t+s)$ (where $s \in \mathbb{R}$).

Let f be a T-periodic function with circular frequency ω. Define the function g such that

$$g(t) = f(st)$$

where $s > 0$. We have

$$g(t + \frac{T}{s}) = f(s(t + \frac{T}{s})) = f(st + T) = f(st) = g(t),$$

so $\frac{T}{s}$ is a period of g. The related circular frequency of g is $\frac{2\pi}{T/s} = \frac{2\pi s}{T} = \omega s$. Call

$$T' = \frac{T}{s} \text{ and } \omega' = \omega s.$$

The exponential Fourier series for $g(t)$ is

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} c'_n e^{jn\omega' t},$$

with the coefficients c'_n given by

$$c'_n = \frac{1}{T'} \int_0^{T'} g(t) e^{-jn\omega' t}\, dt = \frac{s}{T} \int_0^{T/s} f(st) e^{-jn\omega st}\, dt.$$

Let us perform a change of variables by setting $x = st$. Now $dx = sdt$, and the new lower limit of integration is 0 and the upper one T. Therefore, letting c_n denote the coefficients in the exponential Fourier series of f, we have

$$c'_n = \frac{s}{T} \int_0^{T} f(x) e^{-jn\omega x} \frac{1}{s}\, dx = \frac{1}{T} \int_0^{T} f(x) e^{-jn\omega x}\, dx = c_n.$$

Thus $c'_n = c_n$, so time scaling does not change the coefficients c_n, while it *does* change the exponential coefficients from $e^{jn\omega t}$ to $e^{jn\omega' t} = e^{jn\omega st}$, as we saw above. Altogether, we have

$$\boxed{\begin{gathered}\hat{g}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega st} = \hat{f}(st)\\[2em] \text{for } g(t) = f(st), \text{ with each } c_n \text{ denoting a coefficient of } \hat{f}.\end{gathered}}$$

Let us then investigate how time shifting affects Fourier series. Consider the function

$$g(t) = f(t+s).$$

Note that s can be negative, so this can be a time shift to the right as well as left.

The period and the circular frequency do not change in a time shift, so we can let T and ω denote the period and circular frequency of both f and g. Let c_n denote the coefficients of the Fourier series $\hat{f}$ of f, so we have

$$\hat{f}(t) \;=\; \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}.$$

Let us find the Fourier series

$$\hat{g}(t) \;=\; \sum_{n=-\infty}^{\infty} c'_n e^{jn\omega t}.$$

The coefficients c'_n are given by

$$c'_n = \frac{1}{T}\int_0^T g(t)e^{-jn\omega t}\,dt = \frac{1}{T}\int_0^T f(t+s)e^{-jn\omega t}\,dt.$$

We make a change of variables by setting $x = t+s$. Then $dx = dt$, and the lower limit of integration becomes s and the upper one $T+s$. Therefore we have

$$c_n' = \frac{1}{T}\int_s^{T+s} f(x)e^{-jn\omega(x-s)}\,dx = e^{jn\omega s}\frac{1}{T}\int_s^{T+s} f(x)e^{-jn\omega x}\,dx = e^{jn\omega s}c_n.$$

Therefore

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} e^{jn\omega s}c_n e^{jn\omega t}.$$

Note that thereby we have

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} e^{jn\omega s}c_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega(t+s)} = \hat{f}(t+s).$$

That is,

$$\hat{g}(t) = \hat{f}(t+s).$$

Altogether, we have

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} e^{jn\omega s}c_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega(t+s)} = \hat{f}(s+t)$$

for $g(t) = f(t+s)$, with each c_n denoting a coefficient of $\hat{f}$.

Let us also find the trigonometric form of $\hat{g}$. Therefore we let

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\Big)$$

and aim to find what a_n' and b_n' are in

$$\hat{g}(t) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\Big(a_n'\cos(n\omega t) + b_n'\sin(n\omega t)\Big).$$

By the analysis of the exponential series, we have

$$\hat{g}(t) = \hat{f}(t+s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\big(n\omega(t+s)\big) + b_n \sin\big(n\omega(t+s)\big) \right).$$

Using the trigonometric identities

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b), \\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b), \end{aligned}$$

we thus conclude, by minor rearranging, that

$$\begin{aligned} \hat{g}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(& \underbrace{\big(\cos(n\omega s)a_n + \sin(n\omega s)b_n\big)}_{=\, a_n'} \cos(n\omega t) \\ & + \underbrace{\big(\cos(n\omega s)b_n - \sin(n\omega s)a_n\big)}_{=\, b_n'} \sin\big(n\omega t\big) \Big). \end{aligned}$$

Therefore time shift modifies the coefficients a_n and b_n of the Fourier series to the above form. Summing up, we have

$$\boxed{\begin{aligned} &\hat{g}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(\big(\cos(n\omega s)a_n + \sin(n\omega s)b_n\big) \cos(n\omega t) \\ &\quad + \big(\cos(n\omega s)b_n - \sin(n\omega s)a_n\big) \sin\big(n\omega t\big) \Big) \\ &\text{for } g(t) = f(t+s), \text{ with } a_n \text{ and } b_n \text{ denoting coefficients of } \hat{f}. \end{aligned}}$$

3.5 Convergence of Fourier series

Here we look at convergence properties of Fourier series. The related theory is deep, with many open problems, so we can only scratch the surface of the matter instead of giving a complete characterization of convergence behaviour. The point is to understand some of the most important issues related to convergence of typical Fourier series.

The accuracy of the approximation of a function f by a truncated Fourier series, i.e., a finite partial sum, can be approximated by understanding the rate (or speed) of convergence of the Fourier series $\hat{f}$. The rate of convergence bears a close relation to how fast the *sequence* of terms

$$(a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

in the series converges as n tends to infinity. Convergence of this sequence, however, does of course not suffice to guarantee convergence of the *series*.

As the sine and cosine functions get their values from $[-1, 1]$, it is many cases quite reasonable that the convergence rate of the coefficients a_n and b_n is of central importance in determining how fast the overall convergence of the terms $(a_n \cos(n\omega t) + b_n \sin(n\omega t))$ and the series occurs. Understanding how the coefficients behave indeed often helps in proofs of convergence of Fourier series.

Let us define some concepts that will help us analyze convergence rates of general sequences. Let $\mathbb{R}_{\geq 0}$ be the set of nonnegative real numbers. Consider two functions $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, i.e., f and g are functions with domain $\mathbb{N}$ that get their values from $\mathbb{R}_{\geq 0}$. Thereby one can think of f and g as sequences

$$x_0, x_1, x_2, \ldots$$

of nonnegative reals.

We write

$$f \in O(g)$$

if there exists some $m \in \mathbb{N}$ and some nonnegative real number c such that we have

$$f(n) \leq c \cdot g(n) \text{ for all } n > m.$$

Intuitively, this means that *from some large enough number m onwards, the function f always gets values that are at most $c \cdot g$.* Very informally, this means that g in some sense dominates f, at least from some point onwards, as we approach infinity.

Example 3.10. Consider the sequence

$$\left(\frac{5}{n}\right)_{n=1}^{\infty} = \frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \frac{5}{5}, \frac{5}{6}, \dots$$

Define $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ according to this sequence by setting

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \dfrac{5}{n} & \text{if } n \geq 1. \end{cases}$$

Define $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ similarly according to the sequence

$$\left(\frac{1}{n}\right)_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$$

by setting

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ \dfrac{1}{n} & \text{if } n \geq 1. \end{cases}$$

Now, we clearly have

$$g \in O(f).$$

We can here simply write this as follows:

$$\left(\frac{1}{n} \right)_{n=1}^{\infty} \in O\Big(\left(\frac{5}{n} \right)_{n=1}^{\infty} \Big).$$

However, we also have

$$\left(\frac{5}{n} \right)_{n=1}^{\infty} \in O\Big(\left(\frac{1}{n} \right)_{n=1}^{\infty} \Big).$$

It is also clear that we have

$$\left(\frac{5}{n^2} \right)_{n=1}^{\infty} \in O\Big(\left(\frac{1}{n} \right)_{n=1}^{\infty} \Big),$$

while

$$\left(\frac{1}{n} \right)_{n=1}^{\infty} \quad \notin \quad O\Big(\left(\frac{5}{n^2} \right)_{n=1}^{\infty} \Big).$$

Thereby we can intuitively think that $(\frac{1}{n})_{n=1}^{\infty}$ converges *fundamentally slower* than $(\frac{5}{n^2})_{n=1}^{\infty}$. $\square$

Different Fourier series have quite different convergence properties. Roughly, the *smoother the function is*, the faster the rate of convergence. The following theorem formalizes parts of this intuition in relation to convergence of the Fourier coefficients a_n and b_n.

Theorem 3.11. *Let f be a T-periodic function having the Fourier series*

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big) \Big).$$

a) *If the function f is piecewise continuous but not continuous, then $|a_n| \in O(\frac{1}{n})$ and $|b_n| \in O(\frac{1}{n})$.*

b) *If f is continuous but not continuously differentiable, then we have $|a_n| \in O(\frac{1}{n^2})$ and $|b_n| \in O(\frac{1}{n^2})$.*

c) *If f is $k \geq 1$ times continuously differentiable but the $k+1$st derivative is discontinuous, then $|a_n| \in O(1/n^{k+2})$ and $|b_n| \in O(1/n^{k+2})$.*

The proof the theorem is out of the scope of this course.

Example 3.12. The piecewise continuous block wave function of Example 3.3 has jump discontinuities. We showed that its Fourier series is

$$\frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi} \sin\big(nt\big),$$

so $|a_n|$ is zero for $n \geq 1$ while

$$|b_n| = b_n = \begin{cases} 0 & \text{for even } n \\ \dfrac{2}{n\pi} & \text{for odd } n\ . \end{cases}$$

This is clearly in accordance with the item **a)** of Theorem 3.11, as

$$\frac{2}{n\pi} \in \ O(\frac{1}{n}).$$

Now consider the *triangle wave* defined such that

$$g(t) = \begin{cases} t & \text{if } 0 \leq t < \frac{1}{2} \\ -t+1 & \text{if } \frac{1}{2} \leq t < \frac{3}{2} \\ t-2 & \text{if } \frac{3}{2} \leq t < 2, \end{cases}$$

$$g(t+2) = g(t).$$

This function can be shown to have the Fourier series

$$\frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin\big(n\pi t\big),$$

with $a_n = 0$ for all n and thus

$$b_n = \frac{8}{\pi^2 n^2} \begin{cases} (-1)^{(n-1)/2} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases}$$

The triangle wave is continuous but not continuously differentiable, and therefore the fact that

$$|b_n| \leq \frac{8}{\pi^2 n^2}$$

is clearly in accordance with item **b)** of Theorem 3.11. Indeed,

$$\frac{8}{\pi^2 n^2} \in O\left(\frac{1}{n^2}\right)$$

clearly holds. $\square$

3.5.1 The Gibbs phenomenon

In this section we discuss the Gibbs phenomenon, which concerns the behaviour of partial sums of a Fourier series $\hat{f}$ around discontinuities of f. Formal proofs of the results in the section are out of the scope of the course.

Recall the block wave function

1. $$g(t) \;=\; \begin{cases} -\frac{1}{2} & \text{if } \; -\pi \leq t < 0 \\ \frac{1}{2} & \text{if } \; 0 \leq t < \pi. \end{cases}$$

2. $$g(t + 2\pi) = g(t) \text{ for all } t$$

from Example 3.7. We found its Fourier series to be

$$\hat{g}(t) \;=\; \frac{2}{\pi}\Big(\frac{\sin\big(t\big)}{1} + \frac{\sin\big(3t\big)}{3} + \frac{\sin\big(5t\big)}{5} + \;\ldots\; \Big).$$

Let $\hat{g}_{(N)}$ denote the partial sum of $\hat{g}$ that includes all terms up to the terms with coefficients a_N and b_N. Figure 4 shows the partial sums $\hat{g}_{(9)}$ and $\hat{g}_{(19)}$ of $\hat{g}$.

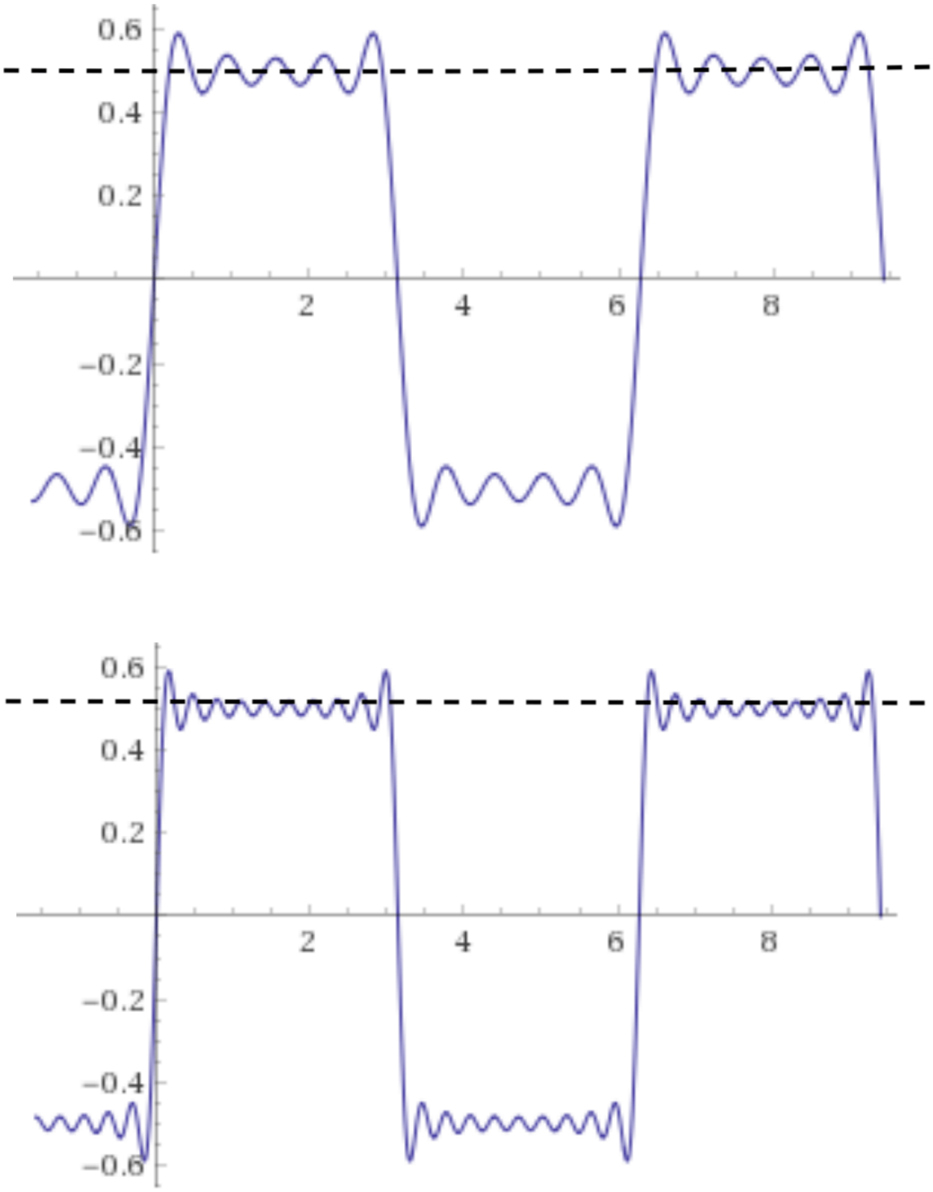

Figure 4: The graphs of the partial sums $\hat{g}_{(9)}$ and $\hat{g}_{(19)}$ of the block wave g from Example 3.7. The dashed line is at $y = \frac{1}{2}$. The "horns" just before and after the discontinuity points are the overshoots and undershoots due to the Gibbs phenomenon.

We notice from Figure 4 that the graphs of $\hat{g}_{(9)}$ of $\hat{g}_{(19)}$ have a clear *overshoot* and *undershoot* close to each point of discontinuity $t = n\pi$ of the function g. See Figure 5 for further details on how partial sums $g_{(N)}$ behave close to discontinuities.

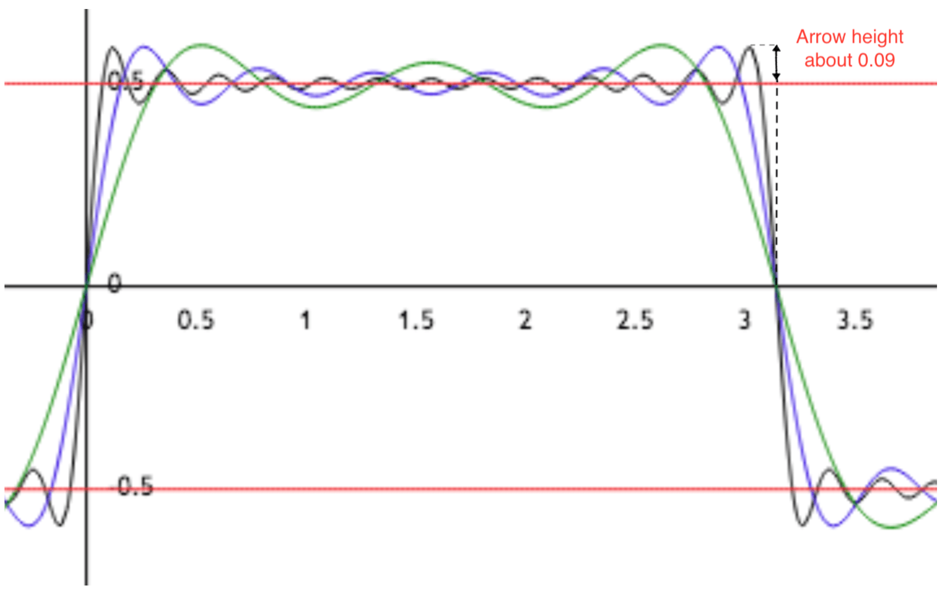

Figure 5: The graphs of the partial sums $\hat{g}_{(5)}$, $\hat{g}_{(11)}$ and $\hat{g}_{(25)}$ in the same figure. Each curve has an overshoot just before the discontinuity of g at $t = \pi$. The overshoot of $\hat{g}_{(5)}$ (the green curve) is quite far from $t = \pi$, the overshoot of $\hat{g}_{(11)}$ (blue curve) somewhat closer, and that of $\hat{g}_{(11)}$ (black curve) the closest. The overshoots get closer and closer to the discontinuity point $t = \pi$ as we consider $g_{(N)}$ for larger and larger values of N. However, the height of the overshoot does not go to zero as N increases. Instead, the height of the overshoot converges to about 9% of the gap $g(\pi^-) - g(\pi^+) = 1$ between the one-sided limits; see the upright double arrow at the top right corner of the figure. There are similar undershoots associated with the one-sided limits at $y = -0.5$. The behaviour is similar for a comprehensive class of periodic functions with jump discontinuities, with the same 9% limit errors at jump discontinuity points. This phenomenon related to overshoots and undershoots at jump discontinuities is known as the Gibbs phenomenon.

The overshoot/undershoot phenomenon shown in the figures is common to a comprehensive class of periodic functions with points of jump discontinuity. It is called the **Gibbs phenomenon**.

We describe the Gibbs phenomenon informally before giving a formal definition. Consider a periodic function f that satisfies the Dirichlet conditions and is *piecewise contiuously differentiable*. Being piecewise continuously differentiable means here that the period of f can be divided into finitely many intervals such that in those intervals, with the possible exception of the endpoints, the function f is differentiable and the derivative is continuous.

Suppose f has a jump discontinuity at t_0. Close to t_0, all partial sums $\hat{f}_{(N)}$ get a value that significantly overestimates/underestimates the value of f. There is either a peak (overestimation, overshoot) or a pit (underestimation, undershoot). The graph of $\hat{f}_{(N)}$ has an overshoot close to t_0 in the vicinity of the greater (meaning greater in the y-direction) one-sided limit of f. Symmetrically, the graph of $\hat{f}_{(N)}$ has an undershoot in the vicinity of the smaller one-sided limit of f. (See Figures 4 and 5 for examples.)

So the partial sums vertically overestimate the value of f where the one-sided limit is anyway higher up in the y-direction and underestimate the value of f where the one-sided limit is lower (in the y-direction).

The points where the undershoot and overshoot are located get horizontally closer and closer to the discontinuity t_0 as we consider sums $\hat{f}_{(N)}$ with greater and greater values of N (cf. Figure 5). However, the size (height) of the overshoot error does *not* tend to zero as $N \to \infty$. The same holds for undershoots. Thus the error between the values of $\hat{f}_{(N)}$ and f does not tend to zero as N increases. But the location of the error shifts closer and closer to the discontinuity point. Because the location of the error shifts,

$$\lim_{N\to\infty} \hat{f}_N(t) = f(t)$$

still holds for all points t where f is continuous.

We then consider the Gibbs phenomenon formally. First, let us define a constant C_{WG}, called the **Wilbraham-Gibbs constant**, as follows:

$$C_{WG} \ = \ \int_0^{\pi} \frac{\sin\left(t\right)}{t}\, dt \approx 1.851937052.$$

Let us also define

$$C = \frac{1}{\pi}C_{WG} - \frac{1}{2} = \frac{1}{\pi}\int_0^{\pi} \frac{\sin\left(t\right)}{t}\, dt - \frac{1}{2} \approx 0.0894898722.$$

Now, to understand the Gibbs phenomenon in a sufficiently general case (from the point of view of applications), consider a function f with period T satisfying the Dirichlet conditions. Suppose f is piecewise continuously differentiable.

Suppose there exists a point of discontinuity t_0 where $f(t_0^+) - f(t_0^-) = d > 0$, so there is a gap of size d at t_0 between the one-sided limits.

Then we have

$$\begin{aligned}
\lim_{N\to\infty} \hat{f}_{(N)}(t_0 + \frac{T}{2N}) &= f(t_0^+) + d \cdot C \\
\lim_{N\to\infty} \hat{f}_{(N)}(t_0 - \frac{T}{2N}) &= f(t_0^-) - d \cdot C.
\end{aligned}$$

Furthermore, for all sequences t_N such that $t_N \to t_0$ as $N \to t_0$, we have

$$\begin{aligned}
\lim_{N\to\infty} \hat{f}_{(N)}(t_N) &\leq f(t_0^+) + d \cdot C \\
\lim_{N\to\infty} \hat{f}_{(N)}(t_N) &\geq f(t_0^-) - d \cdot C
\end{aligned}$$

when the limit on the left hand side of the two equations exist. These four equations imply that close to the discontinuity t_0, the partial sums $\hat{f}_{(N)}$ with very large N will overestimate the vertical gap between the one-sided limits $f(t_0^+)$ and $f(t_0^-)$ by the factor $2C$, i.e., by about 18%. Both sides of the discontinuity contribute about 9% to this error. The first two equations show that such an error will appear around t_0 in large enough partial sums. The last two equations show that no greater error appears around t_0 in large enough partial sums.

Here we assumed that $f(t_0^+) > f(t_0^-)$. In the case $f(t_0^-) > f(t_0^+)$, the analysis is done similarly, in a symmetric way. This time there is an overshoot just before $f(t_0^-)$ and an undershoot right after $f(t_0^+)$.

3.6 Periodic extensions of functions

In this section we consider approximating nonperiodic functions over finite intervals. Before that, we prove the following proposition that is hardly surprising by now.

Proposition 3.13. *Let f be an even function with period T and with Fourier series*

$$\hat{f}(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big).$$

where $\omega = \frac{2\pi}{T}$. Then $b_n = 0$ for all n, so in fact

$$\hat{f}(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\big(n\omega t\big).$$

Similarly, let h be an odd function with period T and with Fourier series

$$\hat{h}(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big)\big).$$

where $\omega = \frac{2\pi}{T}$. Then $a_n = 0$ for all n, so we have

$$\hat{h}(t) \;=\; \sum_{n=1}^{\infty} +b_n \sin\big(n\omega t\big).$$

Proof. Let us first show that the terms b_n in the Fourier expansion of f are all zero. We have

$$b_n \;=\; \frac{2}{T}\int_{d}^{d+T} f(t)\sin(n\omega t)\,dt.$$

By Theorem 2.11, we can shift the interval of integration, whence

$$b_n \;=\; \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt.$$

As a product of an even and an odd function, $f(t)\sin(n\omega t)$ is odd, so this is an integral of an odd function over a symmetric interval, so indeed $b_n = 0$.

Let us then show that the terms a_n are zero for all n in the Fourier series of h. We have

$$a_n = \frac{2}{T}\int_{d}^{d+T} h(t)\cos(n\omega t)\,dt.$$

Again by Theorem 2.11, we can shift the interval of integration, whence

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} h(t)\cos(n\omega t)\,dt.$$

Now, as h is odd, $h(t)\cos(n\omega t)$ is odd (including the special case where $n = 0$). Thus this is an integral of an odd function over a symmetric interval. Thus $a_n = 0$. $\square$

Fourier series of the form

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\big(n\omega t\big)$$

are called **cosine series expansions**, and Fourier series of the form

$$\hat{f}(t) = \sum_{n=1}^{\infty} n_n \sin\big(n\omega t\big)$$

are **sine series expansions**. Thereby Fourier series of even functions are cosine series expansions, while Fourier series of odd functions are sine series expansions.

We are now ready to investigate the issue of approximating nonperiodic functions with Fourier series over intervals of finite length.

Consider a possibly nonperiodic function f. Suppose we are interested mainly in the behaviour of the function inside an interval $(0, T)$. Consider the following questions.

1. Is it still possible to define a useful Fourier series expansion for f somehow, even though f is not necessarily periodic?
2. If yes, is there some unique, best way of doing this?

The answer to the first question is yes. Since we are mainly interested in the behaviour of f inside the period $(0, T)$, we can define a new function g that behaves like f on $(0, T)$ but is forced to be periodic. There are many natural ways of doing this, so the answer to the second question is no. However, the different ways have different pros and cons, so it is good news that we can find several Fourier series expansions that approximate f over $(0, T)$.

Case 1. Basic periodic extensions

Define a function $g(t)$ as follows.

1. $g(t) \ = \ f(t)$ for all $t \in (0, T)$.
2. $g(t+T) \ = \ g(t)$ for all $t \in T$.

Now g is clearly periodic and agrees with f over $(0, T)$. As long as g is regular enough, for example due to satisfying the Dirichlet conditions, we can find a Fourier series for g that approximates f at least over $(0, T)$. However, there are alternative approaches that achieve a similar effect. We next consider two of those.

Case 2. Even periodic extensions

Define a function $g(t)$ as follows.

$$g(t) = \begin{cases} f(t) & t \in [0, T), \\ f(-t) & t \in (-T, 0), \end{cases}$$

$$g(t + 2T) = g(t).$$

Now g is periodic with period $2T$, and again g agrees with f over $(0, T)$. As g is even, the Fourier series of g is a cosine series expansion, i.e., a series of the form

$$\hat{g}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \, cos(n\omega t).$$

Case 2. Odd periodic extensions

Define a function $g(t)$ as follows.

$$g(t) = \begin{cases} f(t) & t \in [0, T), \\ -f(-t) & t \in (-T, 0), \end{cases}$$

$$g(t + 2T) = g(t).$$

The function g is a periodic function with period $2T$, and g agrees with f over $(0, T)$. As now g is an odd function, the Fourier series of g is of the form

$$\hat{g}(t) = \sum_{n=1}^{\infty} b_n \; \sin(n\omega t),$$

that is, $\hat{g}$ is a sine series expansion.

The basic, odd and even periodic extensions are the most common extensions for approximating a nonperiodic function over a period with finite length. Variants of these are easy to design.

3.7 A further form for Fourier series

In this section we present yet another form for Fourier series. We know (cf. Equations 32, 33, 34) that the coefficients c_n of the exponential series relate to the coefficients a_n and b_n of the trigonometric series as follows.

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{a_n - jb_n}{2} \quad \text{for } n = 1, 2, 3 \ldots$$

$$c_n = \frac{a_{-n} + jb_{-n}}{2} \quad \text{for } n = -1, -2, -3 \ldots$$

Note indeed that c_{-n} is the complex conjugate of c_n. We have

$$\left|c_n\right| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \left|c_{-n}\right| \tag{35}$$

and

$$\arg\left(c_n\right) = -\arg\left(c_{-n}\right).$$

We write $\arg(c_n) = \theta_n$. Thus we have

$$c_n = \left|c_n\right| e^{j\theta_n} \quad \text{and} \quad c_{-n} = \left|c_n\right| e^{-j\theta_n}.$$

Thereby we find the following new form for Fourier series:

$$
\begin{aligned}
\hat{f}\left(t\right) &= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\omega t} + c_{-n} e^{-jn\omega t} \right) \\
&= c_0 + \sum_{n=1}^{\infty} \left(\left|c_n\right| e^{j\theta_n} e^{jn\omega t} + \left|c_n\right| e^{-j\theta_n} e^{-jn\omega t} \right) \qquad (36) \\
&= c_0 + \sum_{n=1}^{\infty} \left|c_n\right| \left(e^{j\left(n\omega t+\theta_n\right)} + e^{-j\left(n\omega t+\theta_n\right)} \right) \\
&= c_0 + \sum_{n=1}^{\infty} \left|c_n\right| 2\cos\left(n\omega t + \theta_n\right).
\end{aligned}
$$

Thus we have

$$
\boxed{\hat{f}\left(t\right) = c_0 + \sum_{n=1}^{\infty} 2\left|c_n\right| \cos\left(n\omega t + \theta_n\right), \qquad (37)}
$$

where $\theta_n = \arg\left(c_n\right)$.

The following table summarizes the forms of Fourier series we have identified.

$$
\begin{aligned}
\hat{f}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\big(n\omega t\big) + b_n \sin\big(n\omega t\big) \Big) \\
&= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \\
&= c_0 + \sum_{n=1}^{\infty} \big(\, c_n e^{jn\omega t} + c_{-n} e^{-jn\omega t} \,\big) \\
&= c_0 + \sum_{n=1}^{\infty} 2\big|c_n\big| \cos\left(n\omega t + \theta_n\right) \qquad \text{with } (\theta_n = \arg\big(c_n\big))
\end{aligned}
$$

where

$$a_n = \frac{2}{T} \int_d^{d+T} f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T} \int_d^{d+T} f(t)\sin(n\omega t)\,dt$$

$$c_n = \frac{1}{T} \int_d^{d+T} f(t)e^{-jn\omega t}\,dt$$

and

$$c_n = \frac{a_n - jb_n}{2} \text{ for } n \geq 0, \quad c_n = \frac{a_{-n} + jb_{-n}}{2} \text{ for } n < 0$$

$$2\big|c_n\big| = \sqrt{a_n^2 + b_n^2}\,.$$

The representation

$$\hat{f}(t) = \frac{a_0}{2} \ + \ \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \, \cos\left(n\omega t + \theta_n\right)$$

of the Fourier series of f gives one way of understanding the principal informal intuition behind Fourier analysis. If you think of the terms $\cos\left(n\omega t + \theta_n\right)$ as basis vectors representing higher and higher frequencies, then the amplitudes $\sqrt{a_n^2 + b_n^2}$ represent the amount how much that frequency is present in the function f. Thus the Fourier series $\hat{f}$ can be considered to be the representation of f in terms of basis vectors, specifying for each circular frequency $n\omega$ the scalar $\sqrt{a_n^2 + b_n^2}$ that multiplies the related basis vector $\cos\left(n\omega t + \theta_n\right)$. This is entirely analogous to the constructions in elementary linear algebra where we can *decompose* a complicated vector v into a representation in terms of simple basis vectors, thereby obtaining a better picture of how v is built. Vectors given explicitly in terms of basis vectors are easy to understand and manipulate, so such decompositions are highly advantageous indeed. We shall put this informal idea into use in the coming sections.

3.8 Parseval's theorem

In this section we look at some results concening average values of squared periodic functions $(f(t))^2$. There are several ways of motivating the study of average values of functions $(f(t))^2$ from the point of view of applications, and we will briefly consider this issue below. The main result of the section is *Parseval's theorem* which states that the average of $(f(t))^2$ can be obtained by summing all numbers $|c_n|^2$. This has several important interpretations related to applications; we will *very* briefly discuss at those as well. However, our focus here is strictly the mathematical background theory.

Let $H(x) : \mathbb{R} \to \mathbb{R}$ be a function, and consider a finite interval $[a, b]$, where $b > a$. Recall that the *average value* or mean value of f in $[a, b]$ is defined to be

$$\frac{1}{b-a} \int_a^b H(t)\, dt.$$

In applications, the mean of a function often does not quite provide us with the information we are after due to *polarity issues*, i.e., the interplay of positive and negative values.

For example, if $H(x) = \sin(x)$ and $K(x) = 2\sin(x)$ represent spatial, one-dimensional waves in space, their mean value in the interval $[-\pi, \pi]$ represents the mean displacement from $y = 0$, and this mean value is zero for both functions, despite the clear amplitude difference. This reason for the mean values to be the same is that both functions get both positive and negative values. To counter these polarity effects, it is often more informative to study, e.g., the *mean square values*. For the waves $H(x) = \sin(x)$ and $K(x) = 2\sin(x)$, the mean square values in $[-\pi, \pi]$ are the mean values of $(H(x))^2$ and $(K(x))^2$ in $[-\pi, \pi]$. These values can easily be calculated to be $1/2$ and 2 respectively. Now there is a clear difference between the two values. We note that an alternative would be to consider the *mean absolute values*, i.e., the mean values of $|H(x)|$ and $|K(x)|$. However, mean square values are usually favoured since it is typically easier to work with squared functions than absolute values. However, both options—squares and absolute values—often work perfectly fine.

Square and absolute values do give different numerical values, of course, but this is no problem as long as the theory being built is interpreted correctly, taking the particularties of squares/absolute values carefully into account.

Let $H(t) : \mathbb{R} \to \mathbb{R}$ be a function and $T > 0$. The mean value of H over an interval $[d, d+T]$ is given by

$$\frac{1}{T} \int\limits_{d}^{d+T} H(t)\, dt.$$

In many applications, it is often useful to calculate the mean of a product $f(t)g(t)$ of some functions f and g, which is of course given by

$$\frac{1}{T} \int\limits_{d}^{d+T} f(t)g(t)\, dt.$$

If f and g are T-periodic functions, the following result is frequently useful.

Theorem 3.14 (The multiplication theorem)**.** *Let $f(t)$ and $g(t)$ be T-periodic functions with Fourier series*

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \qquad \text{and} \qquad \hat{g}(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega t}.$$

where $\omega = \frac{2\pi}{T}$. Suppose $f(t)$, $g(t)$ and the product $f(t)g(t)$ satisfy the Dirichlet conditions. Then

$$\frac{1}{T} \int_{d}^{d+T} f(t)g(t)\, dt = \sum_{n=-\infty}^{\infty} c_n \gamma_n^* = \sum_{n=-\infty}^{\infty} c_n^* \gamma_n.$$

Proof. We have $g(t) = \hat{g}(t)$ for all t, with the possible exception of isolated points. Thus the integrals of g and $\hat{g}$ over any interval are the same. Thus we have

$$\begin{aligned}
\frac{1}{T}\int_d^{d+T} f(t)g(t)\,dt &= \frac{1}{T}\int_d^{d+T} f(t)\hat{g}(t)\,dt \\
&= \frac{1}{T}\int_d^{d+T} f(t)\left(\sum_{n=-\infty}^{\infty}\gamma_n e^{jn\omega t}\right)dt \\
&= \sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_d^{d+T} f(t)e^{jn\omega t}\,dt\right)\gamma_n \\
&= \sum_{n=-\infty}^{\infty} c_{-n}\gamma_n = \sum_{n=-\infty}^{\infty} c_n^*\gamma_n,
\end{aligned}$$

where we used the property that $c_{-n} = c_n^*$. We still need to prove that

$$\sum_{n=-\infty}^{\infty} c_n\gamma_n^* = \sum_{n=-\infty}^{\infty} c_n^*\gamma_n.$$

This is immediate, as the two sums have exactly the same terms because $c_{-n} = c_n^*$ and $\gamma_{-n} = \gamma_n^*$. □

We need one more definition before we present Parseval's Theorem. The **average power** of a T-periodic function f is defined as

$$\frac{1}{T}\int_d^{d+T}(f(t))^2\,dt.$$

We often use the notation

$$\frac{1}{T}\int\limits_{d}^{d+T} f^2(t)\,dt.$$

for this.[4]

Theorem 3.15 (Parseval's theorem)**.** *Consider a T-periodic function $f(t)$ and suppose $f(t)$ and $(f(t))^2$ satisfy the Dirchlet conditions. Let*

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}, \quad \textit{where } \omega = \frac{2\pi}{T},$$

be the Fourier series of $f(t)$. Then we have

$$\frac{1}{T}\int_{d}^{d+T} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{38}$$

This formula is known as Parseval's equation or Parseval's identity.

Proof. The result is an immediate consequence of the multiplication theorem. □

Since

$$c_n = \frac{a_n - jb_n}{2}, \qquad c_{-n} = \frac{a_n + jb_n}{2} \qquad \text{and} \qquad c_0 = \frac{a_0}{2},$$

we have

[4]It is of course not necessary to always interpret the average power here to relate to the actual notion of power in, say, signal processing or physics. The integral represents radically different measures in different applications.

$$\left|c_n\right|^2 = \left|c_{-n}\right|^2 = c_n c_{-n} = \frac{a_n - jb_n}{2} \cdot \frac{a_n + jb_n}{2} = \frac{a_n^2 + b_n^2}{4}.$$

Thus we have the following form for Parseval's identity:

$$\boxed{\frac{1}{T}\int_d^{d+T} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} \left|c_n\right|^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}.} \tag{39}$$

Equation 38 can be interpreted in many ways. To give one interpretation, consider Equation 39 which is just an alternative formulation of 38. Recall the following form for Fourier series we deduced earlier:

$$\hat{f}(t) = c_0 + \sum_{n=1}^{\infty} 2\left|c_n\right| \cos\left(n\omega t + \theta_n\right). \tag{40}$$

Recall also Equation 35 which states that

$$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}.$$

Thus, recalling that $c_0 = \frac{a_0}{2}$, we have

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\,\cos\left(n\omega t + \theta_n\right). \tag{41}$$

Thereby the term $\frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{a_n^2+b_n^2}{2}$ of Equation 39 is a simply a sum of the *squared amplitudes* (divided by 2) that multiply the *harmonic basis components* $\cos\left(n\omega t + \theta_n\right)$ in the Fourier series given by Equation 41.[5]

[5]For $n = 0$, the harmonic basis component can be defined to be $\frac{1}{\sqrt{2}}$ (while for $n \geq 1$ the harmonic basis components are indeed $\cos\left(n\omega t + \theta_n\right)$). Then we have $\frac{a_0}{2} = \frac{a_0}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ and we thereby obtain the term $\frac{a_0}{4}$ in Equation 39 by squaring the "amplitude" $\frac{a_0}{\sqrt{2}}$ and dividing by 2—exactly the same way as for the terms in the cases where $n \geq 1$. The lack of elegance in the case for $n = 0$ has no deeper signifigance: it could be tamed by adjusting the background definitions somewhat differently.

Thus the equation

$$\frac{1}{T}\int_{d}^{d+T} f^2(t)\,dt \quad = \quad \frac{a_0^2}{4}+\sum_{n=1}^{\infty}\frac{a_n^2+b_n^2}{2}$$

can be interpreted to state that the *average power of f can be decomposed into a sum of components that represent the squared amplitudes (divided by 2) that multiply the harmonic basis components in a Fourier series of f.* It may be informative to point out that in applications, energy and power are indeed typically proportional to squared amplitudes.

Whatever the case, it is always possible (and fundamental) to interpret Parseval's theorem mathematically, as a way of *decomposing the mean of the square of f into a sum of components relating to the amplitudes of a Fourier series of f.* Mean squares are indeed met everywhere due to the fact that they can nicely deal with polarity issues, as described earlier.

The point of Parseval's theorem is not so much to compute the average of $f^2(t)$ using (halved squares of) the amplitudes $\sqrt{a_n^2+b_n^2}$, as this can be done by direct integration. It is more about realizing that indeed, the mean value of $f^2(t)$ *decomposes* into parts related to multipliers $\sqrt{a_n^2+b_n^2}$, each multiplier being associated with a single *harmonic term* $\sqrt{a_n^2+b_n^2}\cos(n\omega t+\theta_n)$ with its *own individual* circular frequency $n\omega$. Thus the average of $f^2(t)$ can be considered to decompose into individual amplitude-based components $\frac{a_n^2+b_n^2}{2}$ for different circular frequencies.

This thinking leads to the definition of an **amplitude spectrum**. We define the amplitude spectrum of a Fourier series to be the two-ways infinite sequence

$$\ldots|c_{-3}|,\ |c_{-2}|,\ |c_{-1}|,\ |c_0|,\ |c_1|,\ |c_2|,\ \ |c_3|\ldots$$

We note that here we write terms $|c_n|$ instead of $2|c_n|$ which was the amplitude in Equation 40. Recalling that $|c_{-n}|=|c_n|$, we also note that the definition of amplitude spectra is somewhat redundant, as it would not be necessary to write the negative part $\ldots|c_{-3}|,\ |c_{-2}|,\ |c_{-1}|$ at all, since all the related information is encoded in the nonnegative part $|c_0|,\ |c_1|,\ |c_2|,\ \ |c_2|\ldots$ However, it is natural to think about amplitude spectra as encoding information in the following Equation

$$\hat{f}(t) \ = \ c_0 + \sum_{n=1}^{\infty} \left(\left|c_n\right|e^{j\theta_n}e^{jn\omega t} + \left|c_{-n}\right|e^{-j\theta_n}e^{-jn\omega t}\right) \tag{42}$$

which follows directly from Equation 36 as $|c_n| = |c_{-n}|$.

We also define the notion of a **phase spectrum** of a Fourier series, which is given by the sequence

$$\cdots - \theta_3,\ -\theta_2,\ -\theta_1,\ \theta_0,\ \theta_1,\ \theta_2,\ \ \theta_3 \ldots$$

where each θ_n is typically chosen from $[-\pi, \pi]$. Again there is redundancy in the definition, as the terms before θ_0 are simply the negated values of the corresponding terms after θ_0. Note indeed that the terms $-\theta_n$ and $|c_{-n}|$ are associated with the multipliers $\left|c_{-n}\right|e^{-j\theta_n}$ in Equation 42 and the terms θ_n and $|c_n|$ with the multipliers $\left|c_n\right|e^{j\theta_n}$.

Amplitude spectra are usually plotted by drawing for each $n \in \mathbb{Z}$ a vertical line from $(n\omega, 0)$ to $(n\omega, |c_n|)$ with a small x at the point $(n\omega, |c_n|)$. Similarly, phase spectra are plotted by drawing a line from $(n\omega, 0)$ to $(n\omega, \frac{n}{|n|}\theta_{|n|})$ with a small x at $(n\omega, \frac{n}{|n|}\theta_{|n|})$. Of course in practice we only plot a finite number of values of spectra.

4 Discrete Fourier transform

4.1 Defining the discrete Fourier Transform

In a typical application based on an experiment, we only have data points—measured at different time instants—with no clear idea about the underlying function producing the data points. We do not even necessarily know if the related function f is periodic. Suppose our data points—or *samples*—are values of f for inputs from an interval $[0, T]$. Suppose we wish to find approximations of some of the Fourier coefficients c_n occurring in a Fourier series that corresponds to f at least over $[0, T]$. One way forward is as follows.

We suppose the function f is after all periodic with the interval $[0, T]$ being a single period. (Even if the function was not periodic, we would ultimately get an approximation that works somehow for at least $[0, T]$.)

Suppose that our data points are evenly distributed over $\big[0, T\big]$ so that the values of the function f (i.e., the measured data points or samples) are given at the following time instants:

$$0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{\big(N-1\big)T}{N}.$$

Thus the data points divide $\big[0, T\big]$ it into N equal size subintervals of length

$$\frac{T}{N}.$$

The coefficients of the exponential Fourier series are given by

$$c_n = \frac{1}{T} \int_0^T f\big(t\big) e^{-jn\omega t} \, dt.$$

We approximate this integral by the *Riemann sum* with the values of the function evaluated at the left boundary of each subinterval. (Recall that Riemann sums are the basis of the limit process via which integrals arise, i.e., the limit process of approximating the area determined by an integral with rectangular shapes. Recap the notion of Riemann sums if necessary.)

The lengths of the subintervals are

$$\Delta t = \frac{T}{N}.$$

Thus the Riemann sum approximation of

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t} \, dt.$$

is given by

$$\sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) e^{-jn\omega \frac{kT}{N}} \frac{T}{N},$$

which we denote by d_n, that is,

$$c_n \approx d_n \; = \; \frac{1}{T} \sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) e^{-jn\omega \frac{kT}{N}} \frac{T}{N} \; = \; \frac{1}{N} \sum_{k=0}^{N-1} f\left(\frac{kT}{N}\right) e^{-jn\omega \frac{kT}{N}}.$$

In order to simplify notation, we define

$$g_k = f\left(\frac{kT}{N}\right) \quad \text{for} \quad k = 0, 1, 2, \ldots, N-1.$$

Replacing ω by $\frac{2\pi}{T}$, we obtain

$$d_n = \frac{1}{N} \sum_{k=0}^{N-1} g_k e^{-jnk \frac{2\pi}{N}} \quad \text{for} \quad n = \; 0, 1, 2, \ldots, N-1.$$

We define

$$G_n = \sum_{k=0}^{N-1} g_k e^{-jnk\frac{2\pi}{N}} \qquad \text{for} \;\; n = 0, 1, 2, \ldots, N-1,$$

whence $d_n = \dfrac{G_n}{N}$.

Thus the sequence

$$(g_0, g_1, g_2, \ldots, g_{N-1}) \qquad = \qquad \big(g_k\big)_{k=0}^{N-1}$$

based on the samples of the function f gives rise to the sequence

$$(G_0, G_1, G_2, \ldots, G_{N-1}) \qquad = \qquad \big(G_n\big)_{n=0}^{N-1},$$

which we call the *discrete Fourier transform* (DFT) of $(g_0, g_1, g_2, \ldots, g_{N-1})$:

> The **discrete Fourier transform** (DFT) converts a sequence $(g_0, g_1, g_2, \ldots, g_{N-1})$ of samples to the sequence $(G_0, G_1, G_2, \ldots, G_{N-1})$ such that
>
> $$G_n = \sum_{k=0}^{N-1} g_k e^{-jnk\frac{2\pi}{N}} \qquad \text{for} \;\; n = 0, 1, 2, \ldots, N-1. \tag{43}$$

Recalling that $d_n = \dfrac{G_n}{N}$, we see that DFT gives a sequence of terms that—if divided by the number N of samples—approximate the Fourier coefficients c_n.

Since DFT is directly related to the coefficients of the Fourier series of the function f we started with, DFT gives us useful information about the function. (Indeed, if we knew all the coefficients c_n, we would know a lot about the function.)

A useful way of thinking about the DFT is that it *changes perspective* from function values (measured at time-points) to Fourier coefficients (associated with different circular frequencies). Thus we are transferring information from the time domain to a frequency domain.

4.1.1 Defining the inverse discrete Fourier transform

We next construct an inverse to the discrete Fourier transform DFT. The resulting operation will be called the *inverse discrete Fourier transform* or IDFT.

Having the sequence $\big(G_n\big)_{n=0}^{N-1}$, we want to find the sequence $\big(g_k\big)_{k=0}^{N-1}$ of samples we started with. We begin by the following observation:

$$\begin{aligned}
\sum_{n=0}^{N-1} G_n e^{jnk\frac{2\pi}{N}} &= \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} g_m e^{-jnm\frac{2\pi}{N}}\right)e^{jnk\frac{2\pi}{N}} \\
&= \sum_{m=0}^{N-1} g_m\left(\sum_{n=0}^{N-1} e^{-jn\frac{2\pi}{N}(m-k)}\right) \\
&= \sum_{m=0}^{N-1} g_m\Big(\sum_{n=0}^{N-1}\Big(e^{-j2\pi\frac{(m-k)}{N}}\Big)^n\Big). \qquad (44)
\end{aligned}$$

We then notice that the indices m and k satisfy the inequality

$$0 \ \le\ \big|m-k\big| \ \le\ N-1$$

which implies that $\frac{(m-k)}{N}$ is an integer *only* if we have $m-k=0$. When $m=k$, then

$$g_m \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}(m-k)} \right)^n \;=\; g_k \sum_{n=0}^{N-1} 1 = g_k N. \tag{45}$$

If $m \neq k$, then $\frac{m-k}{N}$ is not an integer, and thus, with the help of the formula for *geometric series*, we have

$$g_m \sum_{n=0}^{N-1} \Big(\underbrace{e^{-j\frac{2\pi}{N}(m-k)}}_{=\ q\ \neq\ 1} \Big)^n = g_m \sum_{n=0}^{N-1} q^n = g_m \frac{1-q^N}{1-q} = \frac{1-1}{1-q} = 0$$

where we indeed used the fact that any geometric sum $\sum_{n=0}^{n=m} r^n$ with $r \neq 1$ can be calculated from

$$\sum_{n=0}^{n=m} r^n = \frac{1-r^{m+1}}{1-r}.$$

Thereby we have

$$g_m \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}(m-k)} \right)^n \;= 0 \qquad\qquad \text{for } m \neq k$$

and (repeating Equation 45), we also have

$$g_m \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}(m-k)} \right)^n \;= g_k N \qquad\qquad \text{for } m = k.$$

Therefore, continuing Equation 44, we have

$$\begin{aligned}\sum_{n=0}^{N-1} G_n e^{jnk\frac{2\pi}{N}} &= \sum_{m=0}^{N-1} g_m \Big(\sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}(m-k)} \right)^n \Big) \\ &= 0 + 0 + \cdots + N g_k + 0 + \cdots + 0 = N g_k. \end{aligned}$$

Thus

$$\frac{1}{N} \sum_{n=0}^{N-1} G_n e^{jnk\frac{2\pi}{N}} \;=\; g_k.$$

This gives rise to the **inverse discrete Fourier transform** IFDT:

> The inverse discrete Fourier transform IDFT converts a sequence of $(G_0, G_1, G_2, \ldots, G_{N-1})$ complex numbers to a sequence $(g_0, g_1, g_2, \ldots, g_{N-1})$ such that
>
> $$g_k = \frac{1}{N} \sum_{n=0}^{N-1} G_n e^{jnk\frac{2\pi}{N}} \qquad \text{for } k = 0, 1, 2, \ldots, N-1. \tag{46}$$

To contrast IDFT, we repeat the definition of DFT:

> The discrete Fourier transform DFT converts $(g_0, g_1, g_2, \ldots, g_{N-1})$ to $(G_0, G_1, G_2, \ldots, G_{N-1})$ such that
>
> $$G_n = \sum_{k=0}^{N-1} g_k e^{-jnk\frac{2\pi}{N}} \qquad \text{for} \;\; n = 0, 1, 2, \ldots, N-1.$$

4.2 Properties of the discrete Fourier transform

Periodicity

The discrete Fourier transform is periodic in the sense that for all $n \in \mathbb{Z}$,

$$G_{n+N} = \sum_{k=0}^{N-1} g_k e^{-j(n+N)k\frac{2\pi}{N}} = \sum_{k=0}^{N-1} g_k e^{-jnk\frac{2\pi}{N}} \underbrace{e^{-jk2\pi}}_{=1} = G_n \tag{47}$$

implying that

$$G_{n+N} = G_n \text{ for all } n.$$

Note that we are here operating outside the index set $\{0, \ldots N-1\}$ as the subindex $n+N$ of G_{n+N} is not necessarily in $\{0, \ldots N-1\}$. Thus Equation 47 shows that when operating globally in this way, DFT becomes periodic. We can of course define an infinite set of samples from the basic sequence $(g_0, \ldots, g_{N-1})$ by letting $g_{k+N} = g_k$ for all integers k.

Symmetry

The discrete Fourier transform is symmetric in the sense that for all $n \in \mathbb{Z}$,

$$G_{N-n} = G_n^* \ \text{ for all } n \in \mathbb{Z}.$$

Indeed,

$$G_{N-n} = \sum_{k=0}^{N-1} g_k e^{-jk(N-n)\frac{2\pi}{N}} = \sum_{k=0}^{N-1} g_k e^{jnk\frac{2\pi}{N}} e^{-jnN\frac{2\pi}{N}} = \sum_{k=0}^{N-1} g_k e^{jnk\frac{2\pi}{N}} = G_n^*,$$

whence we also have

$$\left|G_{N-n}\right| = \left|G_n\right| \quad \text{and} \quad \arg\left(G_{N-n}\right) = -\arg\left(G_n\right).$$

By periodicity, we moreover have

$$G_{-n} = G_{N-n} = G_n^*.$$

5 Fourier transform

A Fourier series

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \tag{48}$$

provides a discrete, *frequency-based representation* of f. What we mean by this is that Equation 48 essentially defines a function F (to be specified below) that can be seen as an alternative representation for f, and the domain of F is a set of circular frequencies.

Let us indeed specify what F looks like. Let $S = \{\, n\omega \,|\, n \in \mathbb{Z} \,\}$ be a set of circular frequencies. Let $F : S \to \mathbb{C}$ be a function such that $F(n\omega) = c_n$ for each input $n\omega \in S$. We have

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} F(n\omega) e^{jn\omega t}. \tag{49}$$

Therefore the function F gives a *frequency domain representation* of the function f because as long as we have F, we can compute $\hat{f}(t)$ using Equation 49.

As F contains essentially the same information as Equation 48 (since we can construct 48 based on F and vice versa), a Fourier series can be regarded as a conversion of a function f from a time-domain function to a frequency domain function.

If we drop the assumption of periodicity of f, we can still define a function analogous to the function F above. The function in question is obtained as a result of the *Fourier transform* of f. As opposed to the set S which we defined above to have a discrete set of circular frequencies, the domain of the function that the Fourier transform produces has as its domain essentially the continuous set $\mathbb{R}$ of all possible frequencies. The formal definitions are given in the next section.

5.1 Definition of Fourier transform

In this section we give the formal definitions of the *Fourier integral* and *Fourier transform*. However, before that, we first discuss the so-called *Dirichlet conditions the for Fourier integrals.*

5.1.1 Dirichlet conditions for Fourier integrals

Consider a function $h(t)$ that is bounded in every finite interval, that is, for all (a, b), there exists some real numbers m and M such that $m \leq h(t) \leq M$ for all $t \in (a, b)$. The **improper integral** (of the first kind) of a function $h(t)$ from $-\infty$ to ∞ is defined so that

$$\int_{-\infty}^{\infty} h(t)\,dt \;=\; \lim_{b\to-\infty}\int_{b}^{a} h(t)\,dt \;+\; \lim_{c\to\infty}\int_{a}^{c} h(t)\,dt.$$

Here a can be any constant, for example zero. Now, note that for example

$$\int_{-\infty}^{\infty} t\,dt \;=\; \lim_{b\to-\infty}\int_{b}^{a} t\,dt \;+\; \lim_{c\to\infty}\int_{a}^{c} t\,dt = -\infty + \infty,$$

so $\int_{-\infty}^{\infty} t\,dt$ does not exist (as neither of the integrals in the sum exist due to being $-\infty$ and ∞ respectively).

The **Cauchy principal value** of $h(t)$ from $-\infty$ to ∞ is defined as

$$\mathrm{CPV}_{-\infty}^{\infty}\big(h(t)\big) \;=\; \lim_{b\to\infty} \int\limits_{-b}^{b} h(t)\,dt.$$

We note that $\mathrm{CPV}_{-\infty}^{\infty}(t) = 0$, so the Cauchy principal value exists for the function $h(t) = t$.

We have thus demostrated that there exist functions bounded in every finite interval for which the Cauchy principal value CPV exists but the improper integral does not. However, it can be shown that if the improper integral exists for some function f bounded in all finite intervals, then also the CPV exists, and furthermore, the two values are equal.

Often the notation $\int\limits_{-\infty}^{\infty} h(t)\,dt$ stands for CPV, so one must be careful whether this notation means the improper integral or CPV. We will indeed be careful to make the distinction below, even though it often makes no difference whatsoever.

We are now ready to give a definition which provides sufficient conditions for existence of Fourier integrals, as we shall see later on; Fourier integrals are defined and discussed in the next section. The so-called *Dirichlet conditions for Fourier integrals* are defined as follows:[6]

[6] In the definition, a "*finite interval* $[a, b]$" simply means an interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. This excludes infinite intervals such as, for example, $[0, \infty]$.

Definition 5.1 (Dirichlet conditions for Fourier integrals)**.** A function f satisfies the **Dirichlet conditions for Fourier integrals** if the following conditions are met.

1. For every finite interval $[a, b]$,

   (a) f is piecewise continuous in f,

   (b) $[a, b]$ can be partitioned into finitely many successive intervals such that f is monotone on each of the intervals.

2. the improper integral
$$\int_{-\infty}^{\infty} |f(t)|\, dt$$
   converges, i.e., it has a finite value.

5.1.2 Defining Fourier transforms

In this section we sketch a derivation that results in a formula defining the Fourier transform and its inverse.

Consider a function f that is not necessarily periodic or continuous. Let g denote the *symmetric T-periodic extension of f* defined such that

$$\begin{aligned} g(t) &= f(t) \qquad \text{if } t \in (-\frac{T}{2}, \frac{T}{2}] \\ g(t+T) &= t(t). \end{aligned}$$

Thus g agrees with f when $-\frac{T}{2} < t \leq \frac{T}{2}$ and is forced to be T-periodic with the period interval $(-\frac{T}{2}, \frac{T}{2}]$ copied to the regions outside $(-\frac{T}{2}, \frac{T}{2}]$.

The exponential Fourier series of g is given by

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t},$$

where

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(u) e^{-jn\omega_0 u}\, du \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-jn\omega_0 u}\, du \end{aligned}$$

and

$$\omega_0 = \frac{2\pi}{T}.$$

Now, in the alternative Fourier series (see Equation 37) based on cosines, we have

$$\hat{g}(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \theta_n),$$

so the difference between the circular frequencies of the subsequent *harmonic components* (or *harmonic terms*) $2\left|c_n\right|\cos\left(n\omega_0 t+\theta_n\right)$ and $2\left|c_{n+1}\right|\cos\left((n+1)\omega_0 t+\theta_{n+1}\right)$ is

$$(n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T}.$$

Thus it is natural to define $\Delta\omega = \omega_0$.

We have

$$\begin{aligned}
\hat{g}\left(t\right) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(u\right)e^{-jn\omega_0 u}\,du\right)e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(u\right)e^{-jn\omega_0 u}\,du\right)e^{jn\omega_0 t}\ \frac{2\pi}{T}\frac{1}{2\pi} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(u\right)e^{-jn\omega_0 u}\,du\right)\Delta\omega.
\end{aligned}$$

We define

$$G_T\left(\omega\right) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(u\right)e^{-j\omega u}\,du.$$

Thus

$$\hat{g}\left(t\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}G_T\left(n\omega_0\right)\Delta\omega.$$

We let $T\to\infty$ whence $\frac{2\pi}{T} = \omega_0 = \Delta\omega \to 0$. Therefore, noting that we are summing from $-\infty$ to ∞, it follows quite directly that

$$\lim_{\omega_0=\Delta\omega\to 0,\ T\to\infty} \hat{g}(t) = \lim_{\omega_0=\Delta\omega\to 0,\ T\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} G_T(n\omega_0)\Delta\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\left(\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du\right)d\omega$$

where the integrals are CPVs.

Therefore, since $f(t) = \hat{g}(t)$ when t is a point of continuity, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\left(\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du\right)d\omega \tag{50}$$

if f is continuous at t. Note that f is continuous everywhere except for at isolated points. There is a jump discontinuity at the isolated points t where f is not continuous, and there the integral can be shown to be equal to the average of the one-sided limits $f(t^+)$ and $f(t^-)$.

The double integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\left(\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du\right)d\omega$$

is called the **Fourier integral** of f (and we indeed interpret the integrals as CPVs).

Equation 50 giving the Fourier integral should be compared to Equation 26 which gives the exponential form of Fourier series. Equation 50 can be seen as a continous variant Equation 26.

We will next define the Fourier transform and its inverse. The definitions are constructed directly based on Equation 50. The Fourier transform is analogous to the equation

$$c_n = \frac{1}{T}\int_d^{d+T} f(t)e^{-jn\omega t}dt$$

familiar from Fourier series, and the inverse Fourier transform is a reformulation of the Fourier integral in terms of the Fourier transform.

The **Fourier transform** of f is *defined* as follows:

$$\boxed{\mathcal{F}\{f(t)\} = \mathcal{F}\{f\}(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du} \tag{51}$$

where the integral is a CPV. The formula of the Fourier integral gives the definition of the **inverse Fourier transform**:

$$\boxed{\mathcal{F}^{-1}\{F(\omega)\} = \mathcal{F}^{-1}\{F\}(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega} \tag{52}$$

where the integral is CPV.

The following theorem holds:

Theorem 5.2. *Suppose f satisfies the Dirichlet conditions for Fourier integrals. The Fourier integral converges to f at every point where f is continuous. The Fourier integral converges to the average of one-sided limits of f at every point of discontinuity.*

The derivation of the formula for the inverse Fourier transform—a transform that outputs $f(t)$—was carried out above. However, a full proof of the theorem involving a careful analysis of the Dirichlet conditions for Fourier integrals is beyond the scope of this course and thus omitted.

Example 5.3. Let us find the Fourier transform of the function

$$f : f\big(t\big) = H\big(t\big)e^{-at}$$

where $a > 0$ and H is the Heaviside function (see Section 2.7).

Let us first check that the function f satisfies the Dirichlet conditions for Fourier integrals. Firstly, f has only a single discontinuity and clearly every finite interval where f is continuous is monotone. Secondly, the condition involving the improper integral $\int_{-\infty}^{\infty} \big|f\big(t\big)\big|\ dt$ is also satisfied:

$$\begin{aligned} \int_{-\infty}^{\infty} \big|f\big(t\big)\big|\ dt &= \lim_{b\to\infty} \int_{0}^{b} e^{-at}\, dt \\ &= \lim_{b\to\infty} \big(-\frac{1}{a}e^{-ab} + \frac{1}{a}e^{0}\big) \\ &= \frac{1}{a}, \end{aligned}$$

so all the Dirichlet conditions are satisfied. Therefore the Fourier transform exists. It is calculated as follows:

$$\begin{aligned} F\big(\omega\big) = \int_{-\infty}^{\infty} f\big(u\big)e^{-j\omega u}\, du &= \int_{0}^{\infty} e^{-au}e^{-j\omega u}\, du \\ &= \lim_{b\to\infty} \Big/_{0}^{b} \frac{-1}{a+j\omega}e^{-u\big(a+j\omega\big)} \\ &= \lim_{b\to\infty} \frac{-1}{a+j\omega}\Big(e^{-ba}e^{-jb\omega} - e^{0}\Big) \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

Thus the Fourier transform of f is the function $F(\omega) = \frac{1}{a+j\omega}$. $\square$

It is worth noting that the second requirement of the Dirichlet conditions, namely, that the improper integral

$$\int_{-\infty}^{\infty} \left| f(t) \right| dt$$

converges, is very restrictive. Many frequently occurring functions do not satisfy it. Examples include, e.g., polynomials (including the constant functions other than $y = 0$) and trigonometric functions. However, in practical applications, functions (considered to represent, e.g., signals) are typically finite, meaning that they do not continue to the positive or negative infinity. Then the integral converges.

5.2 Spectra of non-periodic functions

Recall the notions of amplitude spectra and phase spectra from the end of Section 3.8. Spectra provided a complete specification of Fourier series, as they contain enough information for constructing a related Fourier series (as long as we have a given value of ω) simply by plugging them into, e.g., the formula

$$c_0 + \sum_{n=1}^{\infty} 2\big|c_n\big| \cos\left(n\omega t + \theta_n\right) \qquad \text{where } \theta_n = \arg\big(c_n\big) \text{ for all } n = 0, 1, 2, \ldots$$

Thus spectra can be seen as a representation of the function $f(t)$. We now define the analogous notions of spectra for non-periodic functions. These spectra can be plotted, as we will see, analogously to the way the discrete amplitude spectra and phase spectra were, this time resulting in continuous functions.

Now, if $F(\omega)$ is a function resulting in from a Fourier transform of some function f, then it is complex valued and we have

$$\boxed{F\big(\omega\big) = \big|F\big(\omega\big)\big| e^{j \arg\big(F(\omega)\big)}.}$$

The functions $\big|F\big(\omega\big)\big|$ and $\arg\big(F(\omega)\big)$ are, respectively, the **amplitude spectrum** and **phase spectrum** of the function f such that $\mathcal{F}\big\{f\big(t\big)\big\} = F\big(\omega\big)$. While F takes reals as inputs and outputs complex numbers, the functions $\big|F\big(\omega\big)\big|$ and $\arg\big(F(\omega)\big)$ simply map reals to reals. Thus they can be plotted like any function $g : \mathbb{R} \to \mathbb{R}$.

Example 5.4. Recall from above that

$$\mathcal{F}\Big\{H\big(t\big)e^{-at}\Big\}\big(\omega\big) = F\big(\omega\big) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2}.$$

Letting $a = 3$, and recalling that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ for complex numbers, we get

$$
\begin{aligned}
F(\omega) &= \frac{3}{3^2+\omega^2} - \frac{\omega}{3^2+\omega^2}\, j \\
&= \frac{1}{\sqrt{3^2+\omega^2}}\; e^{\,j(-\arctan(\omega/3))},
\end{aligned}
$$

so the amplitude spectrum $|F(\omega)|$ is $\frac{1}{3^2+\omega^2}$ and the phase spectrum $\arg(F(\omega))$ is $-\arctan(\omega/3)$. These two functions are ordinary real-valued functions of a real variable and thereby easy to plot.

It is worth stressing that $|F(\omega)|$ and $\arg(F(\omega))$ are representations of $f(t)$ as frequency domain functions, the function $f(t)$ itself being a time domain function. Thus the Fourier transform and inverse Fourier transforms allow us to change perspectives between the time domain and frequency domain when analysing f. $\square$

5.3 Properties of Fourier transforms

In this section we investigate properties of the Fourier transform. We let

$$\mathcal{F}\Big\{f(t)\Big\}(\omega) = F(\omega) \text{ and } \mathcal{F}\Big\{g(t)\Big\}(\omega) = G(\omega) \text{ with } t, \omega \in \mathbb{R}.$$

We also let $a, b \in \mathbb{C}$.

Linearity

We have

$$\begin{aligned}\mathcal{F}\Big\{af(t) + bg(t)\Big\}(\omega) &= \int_{-\infty}^{\infty} \Big(af(u) + bg(u)\Big)e^{-j\omega u}\,du \\ &= a\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du + b\int_{-\infty}^{\infty} g(u)e^{-j\omega u}\,du \\ &= aF(\omega) + bG(\omega).\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}\Big\{af(t) + bg(t)\Big\}(\omega) = aF(\omega) + bG(\omega).}$$

Scaling

Let $c \neq 0$ and $d > 0$ be real numbers. Recalling the definition of the signum function $\operatorname{sgn}(t)$ from Example 2.9, we have

$$
\begin{aligned}
\frac{1}{|c|}F\left(\frac{\omega}{c}\right) &= \frac{1}{|c|}\int_{-\infty}^{\infty} f\big(y\big)e^{-j\frac{\omega}{c}y}\,dy \\
&= \lim_{d\to\infty}\frac{1}{|c|}\int_{-d}^{d} f\big(y\big)e^{-j\frac{\omega}{c}y}\,dy \\
&= \lim_{d\to\infty}\operatorname{sgn}(c)\int_{-d}^{d} f\big(y\big)e^{-j\frac{\omega}{c}y}\frac{1}{c}\,dy \\
&= \lim_{d\to\infty}\operatorname{sgn}(c)\int_{-|cd|}^{|cd|} f\big(y\big)e^{-j\frac{\omega}{c}y}\frac{1}{c}\,dy \\
&= \lim_{d\to\infty}\int_{-cd}^{cd} f\big(y\big)e^{-j\frac{\omega}{c}y}\frac{1}{c}\,dy \\
&= \lim_{d\to\infty}\int_{-d}^{d} f\big(cu\big)e^{-j\omega u}\,du \qquad \text{(change of variables with } y = cu\text{)} \\
&= \int_{-\infty}^{\infty} f\big(cu\big)e^{-j\omega u}\,du \\
&= \mathcal{F}\{f(cu)\}.
\end{aligned}
$$

Therefore we have

$$
\boxed{\mathcal{F}\Big\{f\big(ct\big)\Big\}\big(\omega\big) = \frac{1}{|c|}F\left(\frac{\omega}{c}\right) \qquad c \neq 0.}
$$

In particular, by substituting $c = -1$, we observe that

$$\boxed{\mathcal{F}\Big\{f\big(-t\big)\Big\}(\omega) = F\big(-\omega\big).}$$

Transforms of derivatives

Suppose—or course—that f satisfies the Dirichlet conditions for Fourier integrals, and suppose also that f is differentiable and the Dirichlet conditions for Fourier integrals also hold for f'. Furthermore, suppose also that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f(-t) = 0$. Then we have

$$\boxed{\mathcal{F}\Big\{f'\big(t\big)\Big\}(\omega) = j\omega F\big(\omega\big).}$$

We justify the property as follows.

$$\begin{aligned}
\mathcal{F}\Big\{f'\big(t\big)\Big\}(\omega) &= \int_{-\infty}^{\infty} f'\big(u\big)e^{-j\omega u}\,du \\
&= \lim_{d\to\infty}\int_{-d}^{d} f'\big(u\big)e^{-j\omega u}\,du \\
&= \lim_{d\to\infty}\left(\,\Big/_{-d}^{d} f\big(u\big)\ e^{-j\omega u} - \big(-j\omega\big)\int_{-d}^{d} f\big(u\big)e^{-j\omega u}\,du\,\right)
\end{aligned}$$

where we integrated by parts. Now, since we assumed that we have $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f(-t) = 0$, the term

$$\Big/_{-d}^{d} f\big(u\big)e^{-j\omega u} = f\big(d\big)e^{-j\omega d} - f\big(-d\big)e^{j\omega d}$$

goes to zero as $d \to \infty$. Therefore

$$\mathcal{F}\Big\{f'(t)\Big\}(\omega) = j\omega F(\omega),$$

as required.

Assume that the Dirichlet conditions for Fourier integrals hold for f and its derivatives $f^{(k)}$ for $k = 1, 2, \ldots, n$. Assume also that $f^{(k)}$ for $k = 0, 1, \ldots, (n-1)$ (where $f^{(0)} = f$) are differentiable and satisfy $\lim_{t\to\infty} f^{(k)}(t) = \lim_{t\to\infty} f^{(k)}(-t) = 0$. Applying repeatedly the above reasoning gives

$$\boxed{\mathcal{F}\left\{f^{(n)}(t)\right\}(\omega) = (j\omega)^n F(\omega).}$$

Shifting properties

The standard shifting properties of Fourier transforms are the following:

$$\boxed{\begin{aligned} \mathcal{F}\Big\{f(t-a)\Big\}(\omega) &= e^{-ja\omega}F(\omega) && \text{(time shift)}, \\ \mathcal{F}\Big\{e^{jbt}f(t)\Big\}(\omega) &= F(\omega - b) && \text{(frequency shift)}. \end{aligned}}$$

The time shift property is sometimes called the **time delay property**, and the frequency shift property the **modulation property**.

The time shift formula can be derived as follows:

$$
\begin{aligned}
\mathcal{F}\Big\{f\big(t-a\big)\Big\}(\omega) &= \int_{-\infty}^{\infty} f\big(u-a\big)e^{-j\omega u}\,du \\
&= \int_{-\infty}^{\infty} f\big(y\big)e^{-j\omega(y+a)}\,dy \qquad \text{(change of variables with } u = y+a\text{)} \\
&= \int_{-\infty}^{\infty} f\big(y\big)e^{-j\omega y}e^{-j\omega a}\,dy \\
&= e^{-j\omega a}\int_{-\infty}^{\infty} f\big(y\big)e^{-j\omega y}\,dy \\
&= e^{-j\omega a}F\big(\omega\big).
\end{aligned}
$$

The frequency shift formula is obtained as follows:

$$
\begin{aligned}
\mathcal{F}\left\{e^{jbt}f\big(t\big)\right\}\big(\omega\big) &= \int_{-\infty}^{\infty} e^{jbu}f\big(u\big)e^{-j\omega u}\,du \\
&= \int_{-\infty}^{\infty} f\big(u\big)e^{-j(\omega-b)u}\,du = F\big(\omega-b\big).
\end{aligned}
$$

The convolution property

The **convolution** of $f * g$ of continuous functions f and g is defined to be

$$
(f*g)\big(t\big) = \int_{-\infty}^{\infty} f\big(t-x\big)g\big(x\big)\,dx = \int_{-\infty}^{\infty} f\big(x\big)g\big(t-x\big)\,dx
$$

where the integrals are interpreted as Cauchy principal values. Convolution in the time domain corresponds to multiplication in the frequency domain, and vise versa:

$$
\begin{aligned}
\mathcal{F}\Big\{(f*g)\big(t\big)\Big\}\big(\omega\big) &= F\big(\omega\big)G\big(\omega\big), \\
\mathcal{F}\Big\{f\big(t\big)g\big(t\big)\Big\}\big(\omega\big) &= \frac{1}{2\pi}(F*G)\big(\omega\big).
\end{aligned}
$$

We prove the first formula as follows:

$$
\begin{aligned}
F\big(\omega\big)G\big(\omega\big) &= F\big(\omega\big)\int_{-\infty}^{\infty} g(u)e^{-ju\omega}\,du \\
&= \int_{-\infty}^{\infty} g(u)e^{-ju\omega}F\big(\omega\big)\,du \\
&= \int_{-\infty}^{\infty} g(u)\mathcal{F}\Big\{f\big(t-u\big)\Big\}\,du \qquad \text{(shift property)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)f\big(t-u\big)e^{-jt\omega}\,du\,dt \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(u)f\big(t-u\big)\,du\right)e^{-jt\omega}\,dt \\
&= \int_{-\infty}^{\infty}\Big((f*g)\big(t\big)\Big)e^{-jt\omega}\,dt \\
&= \mathcal{F}\Big\{(f*g)\big(t\big)\Big\}.
\end{aligned}
$$

We omit the proof of the second formula as it is somewhat involved.

Symmetry

The symmetry property for Fourier transforms is given as follows:

$$\boxed{\mathcal{F}\Big\{\mathcal{F}\{f\}(t)\Big\}(\omega) = \mathcal{F}\Big\{F(t)\Big\}(\omega) = 2\pi f(-\omega).}$$

Here f is first transformed to the frequency domain, thereby obtaining $F(\omega)$. Then the Fourier transform F is regarded as a "time domain function $F(t)$" and the transform is applied again.

We now prove the formula. Using the inverse transform formula, we get

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} F(\omega)\, d\omega$$

which implies that

$$2\pi f(-t) = \int_{-\infty}^{\infty} e^{-j\omega t} F(\omega)\, d\omega.$$

This shows, recalling the formula for the (non-inverse) transform operator $\mathcal{F}$, that

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} e^{-ju\omega} F(u)\, du = \mathcal{F}\Big\{F(t)\Big\}(\omega). \tag{53}$$

Note of course that the equality applies only where the functions are continuous. At points of discontinuity, the values can differ.

It is customary to write

$$f(t) \leftrightarrow F(\omega)$$

in order to indicate that $f(t)$ and $F(\omega)$ are a *transformation pair* so that $\mathcal{F}(\{f(t)\} = F(\omega)$. Note that $A \leftrightarrow B$ does not imply $B \leftrightarrow A$.

Other properties of the Fourier transform

We have

$$F(-\omega) = F(\omega)^*.$$

This is justified as follows. First note that

$$F\big(\omega\big) = \int_{-\infty}^{\infty} f\big(u\big)e^{-j\omega u}\, du$$

$$= \int_{-\infty}^{\infty} f\big(u\big)\cos\big(\omega u\big)\, du - j\int_{-\infty}^{\infty} f\big(u\big)\sin\big(\omega u\big)\, du.$$

Therefore

$$F\big(-\omega\big) = \int_{-\infty}^{\infty} f\big(u\big)e^{-j(-\omega)u}\, du$$

$$= \int_{-\infty}^{\infty} f\big(u\big)\cos\big(\omega u\big)\, du + j\int_{-\infty}^{\infty} f\big(u\big)\sin\big(\omega u\big)\, du = F\big(\omega\big)^*.$$

5.4 Parseval's identity

We have shown that the Fourier series of a T-periodic function f satisfies the Parseval's identity

$$\frac{1}{T}\int_{d}^{d+T} f^2\big(t\big)\,dt = \sum_{n=-\infty}^{\infty} \big|c_n\big|^2.$$

There exists a corresponding result for functions that can be non-periodic:

$$\boxed{\int_{-\infty}^{\infty} f^2\big(u\big)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|F\big(\omega\big)\big|^2\,d\omega \qquad \text{(Parseval's identity).}}$$

We now prove the identity. Firstly, using the definition of the Fourier transform, the convolution property and the definition of convolution, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f^2\big(u\big)e^{-j\omega u}\,du &= \mathcal{F}\Big\{f^2\big(t\big)\Big\}\big(\omega\big)\\ &= \frac{1}{2\pi}\big(F * F\big)\big(\omega\big)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F\big(\omega - x\big)F\big(x\big)\,dx.\end{aligned}$$

Substituting $\omega = 0$ to the equation gives

$$\int_{-\infty}^{\infty} f^2\big(u\big)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\big(-x\big)F\big(x\big)\,dx.$$

Since F satisfies the identity $F\big(-x\big) = F\big(x\big)^*$, we have

$$\int_{-\infty}^{\infty} f^2(u)\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega.$$

Note that our earlier version of Parseval's identity related the *average power* of a function to the coefficients of the related Fourier series. In the current case allowing non-periodic functions, the name of the related concept is (total) **energy**[7]

$$\boxed{E = \int_{-\infty}^{\infty} f^2(u)\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega \qquad \text{(Energy of } f\text{)}.}$$

The function

$$\frac{1}{2\pi}\left|F(\omega)\right|^2 \qquad \text{(or alternatively } \left|F(\omega)\right|^2\text{)},$$

is called the *spectral energy density* (of the signal f). Its graph is the related *energy spectrum*.

[7]Note that the notion of *average* is problematic for non-periodic functions. Thus we now talk about *total* quantities. It depends on the application whether the terms *power* and *energy* are sensible from the point of view of the application.

6 Discrete-time Fourier transform

Recall Section 4 where we discussed the discrete Fourier transform DTF. The point was to approximate a Fourier series based on *samples* g_k, i.e., values of the function underlying the Fourier series. It is possible to approximate the Fourier *transform* (rather than the Fourier series) of a function in an analogous way. This leads to the definition of the *discrete-time Fourier transform* (DTFT).

Let $h > 0$. Assume we have a sequence

$$\Big(f(hk)\Big)_{k=-\infty}^{k=\infty} .$$

of equally spaced samples of a possibly non-periodic function f. The number h is referred to as the *sampling period* of the sequence. Our aim is now to estimate the Fourier transform F of f by using the sequence.

We—quite predictably—assume that f satisfies the Dirichlet conditions for Fourier integrals. We also note that in practical applications, the sequence of samples is of course not infinite, but here we do consider the sequence infinite indeed. However, having a large enough finite set of values samples a should do in real life.

The plain old Fourier transform of the function f is the integral

$$F\big(\omega\big) \;=\; \int_{-\infty}^{\infty} f\big(t\big)e^{-j\omega t}\,dt.$$

We consider the related Riemann sum

$$\sum_{k=-\infty}^{\infty} f\big(t_k\big)e^{-j\omega t_k}\Delta t$$

where the time interval Δt is equal to h and the the points t_k are situated at the lower bounds of the time intervals. Indeed, we can write this sum as

$$\sum_{k=-\infty}^{\infty} f\big(kh\big)e^{-j\omega kh}h.$$

This sum gives the **discrete-time Fourier transform** DTFT:

$$\hat{F}(\omega) = h \sum_{k=-\infty}^{\infty} f(kh) e^{-j\omega kh} \qquad \text{(DTFT)}$$

Note that we constructed a function $\hat{F}$ with a non-discrete domain using a discrete sequence of samples.

Example 6.1. Let the sampling period be $h = 1$. Define

$$x(k) = \begin{cases} a^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where $|a| < 1$ and $k \in \mathbb{Z}$. Then we have

$$\hat{X}(\omega) = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k} = \sum_{k=0}^{\infty} a^k e^{-j\omega k} = \sum_{k=0}^{\infty} \left(ae^{-j\omega}\right)^k$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

by the formula for geometric series. Now $\hat{X}$ is of type $\hat{X} : \mathbb{R} \to \mathbb{R}$ with the non-discrete domain $\mathbb{R}$. $\square$

In a real-life application, we begin sampling at some time instant $t = 0$ and end the sampling process later at (or just before) some time point $t = c$. Then we only obtain values in the interval $[0, c)$. Suppose the values of the sampled function are known to be zero outside this interval. Suppose we have N samples with equally spaced time instants. Letting the sampling period be $h = \frac{c}{N}$, we end up with the sequence

$$\left(g_k\right)_{k=0}^{N-1} = \Big(f(0), f(h), f(2h), \ldots, f\big((N-1)h\big)\Big)$$

of samples.

The discrete-time Fourier transform is

$$\hat{F}(\omega) = h\sum_{k=0}^{N-1} f(kh)e^{-j\omega kh} = h\sum_{k=0}^{N-1} g_k e^{-j\omega kh}.$$

The sum $\hat{F}(\omega)$ approximates the Fourier transform $F(\omega)$ of f. Thus we will study the relation of DTFT to the discrete Fourier transform DTF next, as DTF approximates the Fourier *series* of some function that agrees with f in the finite sampling interval.

We observe that

$$\hat{F}\left(\omega + \frac{2\pi}{h}\right) = h\sum_{k=0}^{N-1} g_k e^{-j\left(\omega+\frac{2\pi}{h}\right)kh} = h\sum_{k=0}^{N-1} g_k e^{-j\omega kh}e^{-j2\pi k} = \hat{F}(\omega).$$

This means that $\hat{F}$ is periodic with period

$$\frac{2\pi}{h}$$

while the Fourier transform F that $\hat{F}$ approximates is *not* periodic.

We split the period $\frac{2\pi}{h}$ into N subintervals and take equally spaced samples of $\hat{F}$, thereby obtaining

$$\hat{F}_n = \hat{F}\left(n\frac{2\pi}{hN}\right) = h\sum_{k=0}^{N-1} g_k e^{-j\left(n\frac{2\pi}{hN}\right)kh} = h\sum_{k=0}^{N-1} g_k e^{-jkn\frac{2\pi}{N}} = hG_n$$

where G_n are the terms of the discrete Fourier transform DTF. Thus we have

$$\boxed{\left(\hat{F}_n\right)_{n=0}^{N-1} = \left(hG_n\right)_{n=0}^{N-1}.}$$

Thus the discrete-time Fourier transform inherits the symmetry property of the discrete Fourier transform ($G_{N-n} = G_n^*$). This seems to suggest that therefore at most half of the sequence $\left(\hat{F}_n\right)_{n=0}^{N-1}$ can be a good approximation of F. As already mentioned, also the periodicity of $\hat{F}$ is problematic.

7 Further developments

In this section we discuss range of issues that are relatively relevant to some parts of Fourier theory but somewhat too advanced to allow a *properly detailed* and *mathematically unambiguous* treatment in this introductory course. The section should be regarded as an appetizer for further studies. The issues discussed in this section will not be part of the exam.

On the Dirac delta function (not in exam)

The *Dirac delta*, also known as the *Dirac delta function*, is an operator that is often used in order to simplify integrals. Intuitively, the Dirac delta $\delta(t)$ is a function that is zero everywhere except for at $t = 0$, and furthermore, any integral

$$\int_{-a}^{a} \delta(0)\, dt$$

over an interval $[-a, a]$ containing 0 gives 1, i.e.,

$$\int_{-a}^{a} \delta(0)\, dt \;=\; 1.$$

The problem here is that *there exists no function with the above properties*. This is easy to see, recalling that the properties should be satisfied for *all* intervals $[-a, a]$ with a a positive real.

The Dirac delta is used in physics to model the *density* of an idealized point charge or point mass. (Indeed, what could the density of a *point* particle be, as a point has volume zero?). The operator first appeared in mathematical analysis in various forms in the 1800s, without a properly rigorous definition. The first rigorous theory was given by Bochner in the 1930s. Currently the delta is typically defined as a *generalized function* or a *distribution*—notions we shall not specify in detail. Thus we look at these issues informally.

For our purposes, the Dirac delta δ is an operator that satisfies the following properties.

1. $\delta\big(t-a\big) = 0$ whenever $t \neq a$.

2. If $a \in (-c, c)$, then $\displaystyle\int_{-c}^{c} \delta\big(t-a\big)\,dt = 1$.

3. $\delta\big(at\big) = \dfrac{1}{|a|}\delta\big(t\big)$.

If $a \in (-c, c)$ and f is continuous at $t = a$, then

4. $\displaystyle\int_{-c}^{c} f\big(t\big)\delta\big(t-a\big)\,dt = f\big(a\big)$.

Informally, we can picture a graph for $\delta(t-a)$ which is zero everywhere and infinitely high at $t = a$. Note that property 1 immediately implies that

$$\delta(t) = \delta(-t). \tag{54}$$

The property 4 above is often called the **sifting property**, as intuitively speaking, the integral sifts through the function f, working through all values of t from $-c$ to c, and selects only the value $f(a)$ at $t = a$ as a result of the integral.

Note that we cannot claim that the four above properties of δ define δ. Instead, we are simply listing some properties of δ. A mathematical definition ought to give a unique mathematical object, so simply specifying the four above axioms leaves it open whether there are multiple objects satisfying the axioms. However, the four axioms above suffice for our purposes here, and a detailed definition is omitted.

Example 7.1. Let us integrate $\cos\big(t\big)\delta\left(t-\frac{\pi}{3}\right)$ from -10 to 10. According to the sifting property, we have

$$\int_{-10}^{10} \cos\big(t\big)\delta\left(t-\frac{\pi}{3}\right)\,dt = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}. \quad \square$$

Let us now make some informal observations. First, notice that

$$\int_{-\infty}^{t} \delta(u)\,du = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

Thus this integral looks like the Heaviside function. Furthermore, it is easy to see that if $t \neq 0$, then we have

$$H'(t) = 0.$$

These results informally suggest that it could be the case that $H'(t) = \delta(t)$. The following results—relating to these informal intuitions—can be derived from the fully developed mathematical theory, although we shall not do the formal derivation here:

$$\boxed{\int_{-\infty}^{t} \delta(u)\,du = H(t) \text{ and } H'(t) = \delta(t).}$$

The Dirac delta can be used to expand Fourier theory so that functions not satisfying the Dirichlet conditions for Fourier integrals do still have sensible Fourier transforms.

Example 7.2. Let us find the Fourier transform of

$$\delta(t-a).$$

We have

$$F(\omega) = \int_{-\infty}^{\infty} \delta(u-a)e^{-j\omega u}\,du = e^{-j\omega a}. \quad \square$$

Example 7.3. Let us find the Fourier transform of the constant function

$$f(t) = c$$

where $c \neq 0$. This function does not satisfy the Dirichlet conditions. However, it is possible to proceed by using the symmetry property. Firstly, if

$$g(t) = c\,\delta(t),$$

then the Fourier transform of $g(t)$ is

$$G(\omega) = \int_{-\infty}^{\infty} c\,\delta(u)\,e^{-j\omega u} du = c\,e^{-j\omega\cdot 0} = c.$$

Thus, recalling the symmetry property

$$\mathcal{F}\Big\{\mathcal{F}\{f\}(t)\Big\}(\omega) = \mathcal{F}\Big\{F(t)\Big\}(\omega) = 2\pi f\big(-\omega\big),$$

we have

$$F\big(\omega\big) = \mathcal{F}\Big\{\underbrace{G\big(t\big)}_{=c}\Big\}\big(\omega\big) = 2\pi g(-\omega) = 2\pi\,c\,\delta\big(-\omega\big) = 2\pi\,c\delta\big(\omega\big).$$

(Note that $\delta(\omega) = \delta(-\omega)$ is a consequence of Equation 54.) $\square$

Example 7.4. Let us find the Fourier transform of the cosine function. The cosine function does not satisfy the Dirichlet conditions. However, we can proceed as follows. First, recall Equation 53, which stated that

$$2\pi f\big(-\omega\big) = \int_{-\infty}^{\infty} e^{-ju\omega} F\big(u\big)\,du = \mathcal{F}\Big\{F\big(t\big)\Big\}\big(\omega\big).$$

Thus we have the transformation pairs

$$f(t) \leftrightarrow F(\omega) \quad \text{and} \quad F(t) \leftrightarrow 2\pi f(-\omega).$$

Thereby, recalling the Fourier transform of $\delta(t-a)$ from Example 7.2, we have

$$\delta\big(t-a\big) \leftrightarrow e^{-j\omega a} \quad \text{and} \quad e^{-jta} \leftrightarrow 2\pi\delta\big(-\omega-a\big) = 2\pi\delta\big(\omega+a\big).$$

Thus we have

$$\begin{aligned}
\mathcal{F}\Big\{\cos\big(at\big)\Big\}\big(\omega\big) &= \mathcal{F}\Big\{\frac{e^{jat}+e^{-jat}}{2}\Big\}\big(\omega\big) \\
&= \frac{1}{2}\mathcal{F}\Big\{e^{jat}\Big\}\big(\omega\big)+\frac{1}{2}\mathcal{F}\Big\{e^{-jat}\Big\}\big(\omega\big) \\
&= \pi\Big(\delta\big(\omega-a\big)+\delta\big(\omega+a\big)\Big). \quad \square
\end{aligned}$$

On the discrete Fourier transform DFT (not in exam)

In this section we consider the *fast Fourier transform* FFT which is a fast way of computing the discrete Fourier transform. Our approach is necessarily quite rough and **corners are cut** at several points **without further mention**. A fully detailed algorithmic analysis is beyond the level of this course. Due to the informal nature of the section, the topic will not be part of the exam.

Recall the definition of DFT:

> The discrete Fourier transform DFT converts $(g_0, g_1, g_2, \ldots, g_{N-1})$ to $(G_0, G_1, G_2, \ldots, G_{N-1})$ such that
>
> $$G_n = \sum_{k=0}^{N-1} g_k e^{-jnk\frac{2\pi}{N}} \quad \text{for} \quad n = 0, 1, 2, \ldots, N-1.$$

The DFT is widely used in applications, so it is desirable to be able to compute DFT fast on a computer. Simply using the above sum in a direct and naive way, already the main terms g_k and $e^{-jnk\frac{2\pi}{N}}$ in the sum require us to perform N multiplications and $N-1$ additions for each value of n. Since $n = 0, 1, 2, \ldots, N-1$, this means $O(N^2)$ calculations (recall the definition of O from Section 3.5).

This easily becomes too slow in practice, where N can be huge, in the thousands or even in the millions. However, there exists an algorithm (or several algorithms to be exact), called the *fast Fourier transform* (FFT) that cuts the number of calculations down to $O(N \log_2 N)$. This is a huge improvement, and is often essential for being able to finish up the required calculations in the first place. For example, for $N = 10^9$, if one main operation took a nanosecond, the naive algorithm would imply a running time of over 30 years and the FFT algorithm less than a minute. (Here, in order to get some estimate, we are, inter alia, simply forgetting all factors that could be hidden in the O-notation.)

Early variants of the idea behind the fast algorithm were known already to Gauss in the early 1800s. Currently, perhaps the most widely known FFT algorithm is the Cooley-Tukey algorithm (1960s). It, and its variants, are among the most important numerical algorithms in existence.

The FFT algorithm uses symmetries of $e^{-jnk\frac{2\pi}{N}}$. Define $W_N = e^{-j\frac{2\pi}{N}}$. Then we have

1. Complex conjugate symmetry: $W_N^{n(N-k)} = W_N^{-nk} = (W_N^{nk})^*$.
2. Periodicity in k and n: $W_N^{nk} = W_N^{n(N+k)} = W_N^{(n+N)k}$.

To ***roughly sketch*** a common approach to FFT, suppose $N = 2^m$ for some m. This assumption is made mainly just to simplify the argument below. We have

$$
\begin{aligned}
G_n = \sum_{k=0}^{N-1} g_k W_N^{nk} &= \sum_{k \text{ is even}}^{N-2} g_k W_N^{nk} + \sum_{k= \text{ is odd}}^{N-1} g_k W_N^{nk} \\
&= \sum_{r=0}^{N/2-1} g_{2r} W_N^{n2r} + \sum_{r=0}^{N/2-1} g_{2r+1} W_N^{n(2r+1)} \\
&= \sum_{r=0}^{N/2-1} g_{2r} W_{N/2}^{nr} + W_N^n \sum_{r=0}^{N/2-1} g_{2r+1} W_{N/2}^{nr}
\end{aligned}
$$

since $W_N^2 = (e^{-j2\pi/N})^2 = e^{-\frac{j2\pi}{N/2}} = W_{N/2}$. Looking at the two sums, we see that they both are DFT-sums, but now with a smaller upper limit $N/2-1$ instead of $N-1$. Now, we have halved the upper limit from $N-1$ to $N/2-1$. Thus we go from N samples to two sets of $N/2$ samples.

Define

$$\begin{array}{ccc} \displaystyle\sum_{r=0}^{N/2-1} g_{2r} W_{N/2}^{nr} & + W_N^n & \displaystyle\sum_{r=0}^{N/2-1} g_{2r+1} W_{N/2}^{nr} \\ G_n^{odd} & + & W_N^n G_n^{even}. \end{array}$$

The key is now that when we compute each of $G_0, G_1, G_2, \ldots, G_n$, some calculations start to appear repeatedly due to the symmetries. In particular, the sums G_n^{odd} and G_n^{even} do not have to be computed every time for each n, but they can instead be reused. Furthermore, we went from the sum with N samples to two sums with $N/2$ samples. This can be repeated, starting from the sums with $N/2$ samples, thus ending up with four sums with $N/4$ samples. Repeating this over and over, it is relatively straightforward to show that this ultimately leads to the desired number $O(N \log_2 N)$ of main operations. We skip the full details.